Computational Methods for calculating multiple point spaces of map germs and applications

Aldicio José Miranda

São Carlos, 2018.

Abstract

The multiple point spaces of a map germ f play an important role in the study of its geometry, as well as the topology of the image or discriminant of a stable perturbation of f. I will introduce some algorithms and implementations on Maple and Singular to obtain the definition ideals of such multiple point spaces in the source and in the target. The aim of the minicourse is to introduce the students and researchers to the use of computational methods for studying properties of explicit examples of singularities.

Contents

1 Introduction

The study of the geometry and topology of a stable perturbation of map germs is one of the main goals in singularity theory and in this sense, the multiple point spaces play a fundamental role. The information encoded in these spaces allows us to describe the geometry and the topology of the image or of the discriminant by providing a complete overview on the map. An important tool to study the multiple point spaces in the target is the determination of the Fitting ideals associated to the image or discriminant, however to find these ideals is not easy and the difficulty appears even in low dimensions of target and source. Therefore the use and development of computational methods and its implementations are essential for the continuity in the development of this theory.

The main purpose of this minicourse is to show algebraic tools and techniques of complex singularities theory through the use of computational methods. First we recover from the Singular manual the general concepts that are usual in singularity theory and commutative algebra, for instance, the Milnor and Tjurina numbers of function germs with isolated singularity, primary decomposition of ideals, Groebner and Standard bases, plot curves and surfaces, etc. Moreover, we also include in this minicourse several examples that are useful to illustrating how these commands and algorithms work.

Then the multiple point spaces in the source are defined for corank 1 map germs, we show an explicit description of all stable types in the source and target for corank 1 map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$. This description is done in terms of subschemes of multiple points of a germ f. We present an implementation in Maple and Singular and several examples are given to compute the stable types in the source. The results relating the multiple point spaces with the main properties of finitely determined map germs appear in the sequel, then we use some algorithms and its implementation to calculating these sets in the source.

Last, but not least, we describe the multiple point spaces in the target for map germs which are not necessarily of corank 1. The kth multiple points space $M_k(f)$ is the closure in the image of the set of points having k or more preimages, counting multiplicities, and in this case, $M_k(f)$ is defined by means of the Fitting ideals of a presentation matrix associated to the germ f, concept developed by Mond and Pellikaan. We present an algorithm and a SINGULAR library, to obtain such presentation matrix. The Fitting ideals are very relevant because its tell us a great deal about the geometric behavior of such maps.

The author would thanks the organizing committee of this School on Singularity Theory for this invitation to give this minicourse, and also thank Marcelo Saia for his comments.

2 Introduction to Singular

In this section we shall give a short introduction to the computer algebra system SINGULAR, [5].

Most of the text in this chapter was extracted verbatim from the, [11], [10], and online manual of SINGULAR. For more details we refer to the Singular Manual, which is offered as an online help for SINGULAR.

The SINGULAR, is a Computer Algebra system (CA-system), designed for polynomial computation. Which was developed in order to support mathematical research in commutative algebra, algebraic geometry and singularity theory. The main computational objects of SINGULAR, are ideals and modules over a large variety of baserings. The baserings are polynomial rings or localizations thereof over a field (e.g., finite fields, the rationals, floats, algebraic extensions, transcendental extensions) or over a limited set of rings, or over quotient rings with respect to an ideal.

SINGULAR features one of the fastest and most general implementations of various algorithms for computing Groebner bases. Furthermore, it provides polynomial factorization, resultant, gcd computations, syzygy, free-resolution computations, and many more related functionalities.

SINGULAR is available, free of charge, as a binary programme for most common hardware and software platforms. Release versions of SINGULAR can be downloaded through site https://www.singular.uni-kl.de/download.html. It is also possible to use the official web-interface of SINGULAR based on the InteractiveShell package by Franziska Hinkelmann, Lars Kastner and Mike Stillman, through browser https://www.singular.uni-kl.de:8003/.

The development of Singular started in early 80's in order to support the research in commutative algebra, algebraic geometry and singularity theory. The areas of applications of singular grew significantly. Now it includes symbolic-numerical solving, integer programming, tropical geometry, noncommutative computer algebra, etc.

2.1 Getting Started

Before starting with the first commands in SINGULAR, we define polynomial ring and ideals.

The polynomial ring $K[x_1, \dots, x_n]$ in n variables over ring K is the set of all polynomials together with the usual operations, of sum and multiplications:

$$\sum_\alpha a_\alpha x^\alpha + \sum_\alpha b_\alpha x^\alpha := \sum_\alpha (a_\alpha + b_\alpha) x^\alpha,$$

$$\left(\sum_\alpha a_\alpha x^\alpha\right)\left(\sum_\beta b_\beta x^\beta\right) := \sum_\gamma \left(\sum_{\alpha+\beta} a_\alpha b_\beta\right) x^\gamma.$$

Where, $x = (x_1, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n, a_\alpha, b_\beta \in K$, and $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$. Addition and subtraction of two polynomials are performed by adding or subtracting corresponding coefficients.

$K[x_1,\dots,x_n]$ is a commutative ring with identity $1 = x_1^0 \cdot \dots \cdot x_n^0$, which we identify with the element $1 \in K$. Elements of $K \subset K[x]$ are called constant polynomials. K is called of ground ring of $K[x]$ or ground field if K is a field.

Given a finite collection of polynomials $f_1,\dots,f_s \in K[x_1,\dots,x_n]$, we can create other polynomials dependent on these by multiplying by arbitrary polynomials in $K[x_1,\dots,x_n]$ and taking the sum.

Definition 2.1. *A subset $I \subset \mathbb{K}[x_1,\dots,x_n]$ is an ideal if it satisfies:*

i. $0 \in I$.

ii. *If $f, g \in I$, then $f + g \in I$.*

iii. *If $f \in I$ and $h \in \mathbb{K}[x_1,\dots,x_n]$, then $hf \in I$.*

Definition 2.2. *Let $f_1,\dots,f_s \in \mathbb{K}[x_1,\dots,x_n]$. Then we set $\langle f_1,\dots,f_s\rangle$ to denote the collection*

$$\langle f_1,\dots,f_s\rangle = \{p_1 f_1 + \dots + p_s f_s : p_i \in \mathbb{K}[x_1,\dots,x_n], \quad i = 1,\dots,s\}.$$

An important result that we will not prove here is:

Theorem 2.3 (Hilbert Basis Theorem)**.** *Every ideal $I \in \mathbb{K}[x_1,\dots,x_n]$ has a finite generating set. In other words, given an ideal I, there exists a finite number of polynomials $f_1,\dots,f_s \in \mathbb{K}[x_1,\dots,x_n]$ such that $I = \langle f_1,\dots,f_s\rangle$.*

1. Once SINGULAR is started, it awaits an input after the prompt ">".

2. Every statement has to be terminated by ";" . The semicolon tells the computer that the inputted command is to be interpreted.

3. All objects have a type, e.g., integer variables are defined by the word `int`.

```
> int k = 12;
```

4. An assignment is made using the symbol "=".

5. Test for equality is done using "==".

6. Test for inequality is done using "! =" or "<>", where 0 represents the boolean value FALSE, and any other value represents TRUE.

```
> k == 12;
1;
> k != 12;
0;
```

7. The value of an object is displayed by simply typing its name.

```
> k;
12;
```

8. The command

```
> help;  //open the help file of singular.
```

9. Text starting with // denotes a comment and is ignored in calculations.

Variables of type string can also be defined and are delimited by ´´ (double quotes). They may be used to comment the output of a computation. If a string contains valid Singular commands, it can be executed using the function `execute`. The result is the same as if the commands would have been written on the command line. This feature is especially useful to define new rings inside procedures.

In Singular one can define polynomial rings over the following fields:

1. the field of rational number $\mathbb{Q}$;
2. finite fields $\mathbb{F}_p$, p a prime number ≤ 2147483629;
3. finite fields $GF(p^n)$ with p^n elements, p a prime, $p^n \leq 2^{15}$;
4. transcendental extensions of $\mathbb{Q}$ or $\mathbb{F}_p$;
5. simple algebraic extensions of $\mathbb{Q}$ or $\mathbb{F}_p$;
6. simple precision real floating point numbers;
7. arbitrary prescribed real floating point numbers;
8. arbitrary prescribed complex floating point numbers.

Remark 2.4. Indeed, the computation over the above fields is exact, only limited by the internal memory of the computer. Strictly speaking, floating point numbers, as in 6. – 8., do not represent the field of real (or complex) numbers. Because of rounding errors, the product of two non zero elements or the difference between two unequal elements may be zero. Of course, in many cases one can trust the result, but we should like to emphasize that this remains the responsibility of the user, even if one computes with very high precision.

To perform a calculation in Singular it is first absolutely necessary to define the ring over which one is working.

1. **Computation in the field of rational numbers:**

```
> ring A=0, x, dp;
> number n = 2/3;
> n^3;
8/27;
```

`> ring A=0, x, dp;` define a ring of polynomials in variable x, characteristic zero, with coefficients in $\mathbb{Q}$, and monomial global order `dp`. (`dp` is a degree reverse lexicographical ordering).

2. **Computation in finite fields:**

```
> ring A1=32003, (x), dp;
> number n=1010253;
> n^8;
3451;
```

`> ring A1=32003, x, dp;` define the ring A1= $\mathbb{Z}_p[x]$, polynomial ring of characteristc $p = 32003$ with coefficients in the field $\mathbb{Z}_p$.

3. **Computation with real floating point numbers,** 100 **digits precision:**

```
> ring r=(real,100),(x),dp;
> number n=2/5;
> n^9000;
0.346674542952376686866987520171913752858044459504854910111826094326663985706838293456814864759869368
  85706838293456814864759869368 9e-3581
>
```

We have a number with 3581 digits after point. However, only 100 digits are computed.

4. **Computation with complex floating point numbers,** 20 **digits precision:**

```
> ring R1=(complex, 20,i), (x), dp;
> number n = 123456.0+0.021i;
> n^7;
0.43710463467674779545e+36+i*0.52046381947051861050e+31;
```

The result is a complex number whith real part and imaginary part with respectively 36 and 31 digits. However, only 20 digits are computed.

5. **Computation in polynomial rings (over $\mathbb{Q}$)**

```
> ring R=0,(x,y,z), lp;
> poly f=x4+zy2;
> f*f-2*f;
x8+2x4y2z-2x4+y4z2-2y2z
>
```

$R = \mathbb{Q}[x, y, z]$ is a polynomial ring of characteristic 0 over $\mathbb{Q}$, in variables (x, y, z). (`lp` is a monomial global lexicographic ordering.)

6. **More examples of rings which one can define in SINGULAR.**

   i. `> ring r1=integer,(a,b),lp;`
   The ring of integers, variables a and b, ordering lp.

   ii. `> ring r2=(integer, 20),(a,b),lp;`
   The ring of integers modulo 20, variables a and b.

   iii. `> ring r3=(integer,2,12),(a,b),lp;`
   The ring of integers modulo 2^{12}, variables a and b.

   iv. `> ring r4 = 0,(x(1..48)),dp;`
   The ring over $\mathbb{Q}$, characteristic 0, variables $x(1), \ldots, x(48)$.

   v.
```
> ring r5 = 0,(x,y,z),dp;
> ideal I=x^2,y,z^2;
> qring r6 = std(I);  \\quotient ring modulo I
```
   The ring $r6$ is quotient ring modulo I, $r6 = \frac{\mathbb{Q}[x,y,z]}{\langle x^2,y,z^2\rangle}$.

   vi. `> ring r=(0,a,b),(x,y,z),lp;`
   The ring of polynomials in the variables x, y, z, where the coefficients are rational terms in the variables a and b, $r = \mathbb{Q}(a, b)[x, y, z]$. Important is that the variables in the first brackets can appear in the denominator of fractions, the ones in the second brackets may not.

   vii. For a mixed ordering, we obtain others rings.
   1. We define the ring $(\mathbb{Q}[x, y]_{<x,y>})[z]$, polynomial ring in variable z, with coefficients in the localization of polynomial ring $\mathbb{Q}[x, y]$, by
```
> ring r=0,(z,x,y),(dp(1),ds(2));
> r;
//   characteristic : 0
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    z
//        block   2 : ordering ds
//                  : names    x y
//        block   3 : ordering C
```

   viii. For quasihomogenous input ideals, Groebner bases computations are generally faster with the orderings $\mathrm{Wp}(w_1, \ldots, w_n)$ or $\mathrm{Ws}(w_1, \ldots, w_n)$.
```
> ring A = 0,(x,y,z), Wp(5,3,2);
```

ix. Let R be a ring. Elements of submodules are given by vectors in R^n as generators. The canonical basis elements e_i of R^n $(e_i = (0, \cdots, 1, \ldots, 0), 1$ at place i), are denoted by `gen(i)` in SINGULAR.

```
> ring R=0,(x,y,z),dp;
> matrix M[2][2] = xy,yz,xz,z2;
> print(M);
xy,yz,
xz,z2
> module ker = syz(M);  ker;
ker[1]=x*gen(2)-z*gen(1)
```

ker is the submodule of $\mathbb{Q}[x, y, z]^2$.

x. **Module orderings**

```
> ring R2=0,(x,y,z),(c,dp);
> module ker = imap(R,ker);
> ker;
ker[1]=[-z,x]
```

The ordering (c,..) has the effect that output is represented as component-wise.

In SINGULAR, we can define several rings in the same ambient. Defining a ring makes this ring the current active basering. If the current ring is r, and one want to calculate in the ring $r1$, which has been defined before, this can be done using the function setring.

```
> setring r1;
```

Now, the current ring is $r1$, but the data of the ring r has not been deleted.

In SINGULAR, it is also possible to program, create procedures, libraries, and the syntax is based on the well-known C language. The SINGULAR distribution contains several libraries, each of which contains a collection of procedures, and can be loaded when necessary.

```
> LIB "all.lib"; //loads and lists all libraries of Singular.
```

Objects defined in a ring can be carried to other rings without having to redefine them again.

`imap` is the map between rings and quotient rings (`qring`) with compatible ground fields which is the identity on variables and parameters of the same name and 0 otherwise. `imap` uses the names of variables and parameters and it can map parameters to variables.

`fetch` is the identity map between rings and quotient rings (`qring`), this map uses the position of the ring variables, not their names, i.e. the i-th variable of the source ring is mapped to the i-th variable of the basering. The coefficient fields must be compatible. `fetch` offers a convenient way to change variable names or orderings.

```
> ring r1=0,(x,y,z),dp;
> ring r2=0,(y,x,z),dp;
> ring r3=0,(a,b,c),dp;
> setring r1;
> poly f = x2+y3+z2+xz+y2z;
> f;
y3+y2z+x2+xz+z2
> setring r2;
> poly f=imap(r1,f);f;
y3+y2z+x2+xz+z2                        \\uses the names of variables
> poly g=fetch(r1,f);g;
x3+x2z+y2+yz+z2                        \\uses the position of variables
> setring r3;
> poly f=imap(r1,f);f;
0
> poly g=fetch(r1,f);g;
b3+b2c+a2+ac+c2
```

"Maps" are ring maps from a preimage ring (source) into the basering (target), defined by specifying images for source variables in the target ring. The target of a `map` is always the actual basering. Maps between rings with different coefficient fields are possible.

```
> ring r1=0,(x,y,z),dp;
> ring r3=0,(a,b,c),dp;
> setring r1;
> ideal i=x2+y2+z;
> setring r3;
> map M=r1, a2,b,c+1;
> ideal j=M(i);j;
j[1]=a4+b2+c+1
```

In this example, we have that M is the map:

$$\begin{array}{rcrcc} M & : & r_1 & \longrightarrow & r_3 \\ & & x & \longmapsto & a^2 \\ & & y & \longmapsto & b \\ & & z & \longmapsto & c+1 \end{array}$$

Then, $M(i) = (a^2)^2 + b^2 + c + 1 = a^4 + b^2 + c + 1$.

Grobner and Standard bases, are used for example, in question to decide whether some function vanishes on a variety, or in algebraic terms if a polynomial is contained in a given ideal. For this we calculate a standard bases using the command `groebner`.

```
> LIB "standard.lib";   \\libraries for standard or Groebner bases
> ring r=0,(x,y),lp;
```

```
> ideal I=xy-1,y2-1;
> ideal G=groebner(I); \\return Groebner bases of ideal I
> G;
G[1]=y2-1
G[2]=x-y
```

We can plot curves and surfaces in SINGULAR with the `plot` command in the library surf.lib.

```
LIB "surf.lib";
ring r = 0,(x,y,z),dp;
poly p = z^2-x^2*y; // Whitney umbrella
plot(p);
```

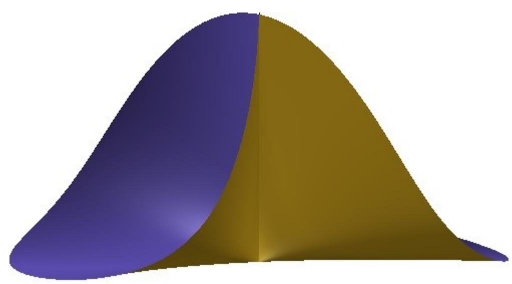


Figure 1: Whitney umbrella.

We can create library in SINGULAR. Next example is a library to calculate the Milnor and Tjurina number of a hypersurface, with isolated singularity.

```
version="$Id: milnortjurinanumber.lib  2018-07-15 $";
category="Singularity Theory, Commutative Algebra";
info="
LIBRARY:  milnortjurinanumber.lib      Compute Milnor and Tjurina number
AUTHORS:  name of author01,       email01@email01.com
          name of author02,       email02@email02.com
PROCEDURES:
  MilnorTjurina(); compute Milnor and Tjurina numbers
    ";
//-----------------------------------------------------
proc MilnorTjurina (poly p)
{
ideal j=jacob(p);
list L=vdim(std(j)), vdim(std(j+p));
print("//List with Milnor and Tjurina number resp.");
return(L);
 }
> poly f = z4+y3+x2+xy;
```

```
> MilnorTjurina(f);
//List with Milnor and Tjurina number resp.
[1]:
   6
[2]:
   3
```

We will now see some very useful commands in SINGULAR.

The command `reduce`, reduces a polynomial, vector, ideal or module to its normal form with respect to an ideal or module represented by a standard bases. Returns 0 if, and only if, the polynomial (resp. vector, ideal, module) is an element (resp. subideal, submodule) of the ideal (resp. module).

```
> ring r=0,(x,y),ds;
> ideal I=x2+y2,xy;
> ideal J=std(I);
> reduce(yx2+y3+x2y,J);
0                          //yx2+y3+x2y is an element of I
> reduce(x+y,J);
x+y                        //x+y is not a element of I
```

The command `kbase`, computes a vector space basis consisting of monomials of the quotient ring by the ideal, respectively of a free module by the module, in case it is finite dimensional and if the input is a standard bases with respect to the ring ordering.

```
> ring r=0,(x,y),ds;
> ideal i=x2,y2;
> vdim(std(i));
4
> kbase(std(i));
_[1]=xy
_[2]=y
_[3]=x
_[4]=1
```

The command `division`, computes a division with remainder for two ideals or two modules. Example: Let f be the polynomial defined by $x^2y + xy^2 + y^2$ and let the ideal $I = \langle y^2 - 1, x - y\rangle$. Find a_1, a_2, r such that $f = a_1(y^2 - 1) + a_2(x - y) + r$, r is the remainder.

```
> ring r=0,(x,y),lp;
> ideal I=y2-1,x-y;
> poly f=x2y+xy2+y2;
> list L=division(f,I);
```

```
> L[1]; //return ai´s
_[1,1]=2x+1
_[2,1]=xy+2
> L[2]; //return remainder r
_[1]=2y+1
```

Therefore, $x^2y + xy^2 + y^2 = (2x+1)(y^2-1) + (xy+2)(x-y) + (2y+1)$.

See below that, `reduce(f,std(I))`, return $r = 2y + 1$.

```
> ring r=0,(x,y),lp;
> ideal I=y2-1,x-y;
> poly f=x2y+xy2+y2;
> reduce(f,std(I));
2y+1                   //equal the remainder above
```

The command `factorize`, computes the irreducible factors as an ideal of the polynomial.

```
> ring r=0,(x,y),dp;
> poly i=(x-1)^2*(y^2+1);
> factorize(i);
[1]:
   _[1]=1
   _[2]=x-1
   _[3]=y2+1
[2]:
   1,2,1      //factor (x-1) has multiplicity 2
              //factor y^2+1 is irreducible in Q[x]

> ring q=(0,i),y,dp;
> minpoly =i2+1;
> poly p=y2+1;
> factorize(p);
[1]:
   _[1]=1
   _[2]=y+(i)
   _[3]=y+(-i)
[2]:
   1,1,1
```

The command `resultant` computes the resultant of two polynomials with respect to one variable.

```
> ring r=0,(x,y,z),dp;
> poly f=x2+y3+xz2+z2+1;
> poly g=xy+x2z+y2;
> resultant(f,g,z);
x4y3+x6+x3y2+2x2y3+xy4+x4+x2y2+2xy3+y4
```

The command `subst`, substitutes one or more ring variable(s) or parameter variable(s) by (a) polynomial(s).

```
> ring r=0,(x,y,z),dp;
> poly f=x2+y3+xz2+z2+1;
> subst(f,x,y,z,1);
y3+y2+y+2
```

2.2 Using Singular

All calculations in a CA-system are performed over the field of rationals numbers or over a field extensions thereof, but not over the field of real or complex numbers. The calculations does not change, but we need to be aware of this when analysing computacional results.

In the following examples, we will use elimination and primary decomposition.

Definition 2.5. *Let $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. The ℓth elimination ideal I_ℓ is the ideal of $\mathbb{K}[x_{\ell+1}, \ldots, x_n]$ defined by*

$$I_\ell = I \cap \mathbb{K}[x_{\ell+1}, \ldots, x_n].$$

I_ℓ is an ideal of $\mathbb{K}[x_{\ell+1}, \ldots, x_n]$ and different orderings of variables results in different elimination ideals. Groebner bases allow us to eliminate the variables $x_1, \ldots, x_\ell$.

Theorem 2.6 ([3], Theorem 2, (The Elimination Theorem))**.** *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal and let G be a Groebner bases of I with respect to lex order where $x_1 > \ldots > x_n$. Then, for every $0 \leq \ell \leq n$, the set*

$$G_\ell = G \cap \mathbb{K}[x_{\ell+1}, \ldots, x_n]$$

is a Groebner bases of the ℓth elimination ideal I_ℓ.

Intersection with subrings or elimination of variables is one of the most important applications of Groebner bases. In the case of polynomial ring, we need a global elimination ordering for the variables.

Example 2.7. Let $f : \mathbb{K}^2 \to \mathbb{K}^3$ be a map defined parametrically by $f(x, z) = (x, -4z^3 - 2xz, -3z^4 - xz^2)$. Compute an implicit equation for this surface using the Singular.

```
> ring r=0,(x,z,X,Y,Z),dp;
> ideal I=X-x, Y-(-4z^3-2xz), Z-(-3z^4-xz^2);
> ideal eI=eliminate(I,xz);
> eI;
eI[1]=4X3Y2+16X4Z+27Y4+144XY2Z+128X2Z2+256Z3
> short=0;
> eI;
eI[1]=4*X^3*Y^2+16*X^4*Z+27*Y^4+144*X*Y^2*Z+128*X^2*Z^2+256*Z^3
```

Example 2.8. Compute the singular locus of eI obtained in the Example 2.7. We can apply the Jacobian criterion, since the ideal eI is equidimensional.

```
> ring R=0,(X,Y,Z),dp;
> ideal J=imap(r,eI);
> J;
J[1]=4X3Y2+16X4Z+27Y4+144XY2Z+128X2Z2+256Z3
> dim(std(J));
2
> ideal J2=slocus(J);
> J2;
J2[1]=4X3Y2+16X4Z+27Y4+144XY2Z+128X2Z2+256Z3
J2[2]=16X4+144XY2+256X2Z+768Z2
J2[3]=8X3Y+108Y3+288XYZ
J2[4]=12X2Y2+64X3Z+144Y2Z+256XZ2
```

When solving a system of equations, it is important to understand the geometry of the zero-set. To better visualise the set of zeros of ideal `J2`, we can try to decompose ideal `J2`. An important step towards this goal is finding the primary decomposition of the defining ideal. For decomposing an algebraic variety into its irreducible components, the algebraic foundation for this is given by primary decomposition of ideals. Primary decomposition provides a generalization of the factorization of an integer as a product of prime-powers. In some sense, a prime ideal in a ring A is a generalization of a prime number. The corresponding generalization of a power of a prime number is a primary ideal.

Let A be a ring. An ideal $I \subsetneq A$ is primary if $ab \in I$ imply either $a \in I$ or $b^n \in I$ for some $n > 0$. Cleary every prime ideal is primary.

Theorem 2.9. *Let A be a Noetherian ring and $I \subsetneq A$ be an ideal, then there exists an irredundant decomposition $I = \cap_{i=1}^{r} Q_i$ of I, where Q_i, $i = 1, \dots, r$ are primary ideals.*

There are two algorithms implemented in Singular which provide primary decomposition of an ideal, implemented in the library primdec.lib, `primdecGTZ`, algorithm based on Gianni-Trager-Zacharias, written by Gerhard Pfister and `primdecSY`, based on Shimoyama-Yokoyama, written by Wolfram Decker and Hans Schoenemann. The result of these two commands is returned as a list of pairs of ideals, where the first ideal corresponding primary ideal and the second ideal is the prime ideal.

```
> list L=primdecSY(J2);
> L;
[1]:
   [1]:                                      //the 1st primary component
      _[1]=9XY2+40X2Z-96Z2
      _[2]=2X3+27Y2+72XZ
      _[3]=27Y4-512X2Z2+2048Z3
   [2]:                                      //the 1st prime component
      _[1]=9Y2+32XZ
```

```
      _[2]=X2-12Z
[2]:
   [1]:                                  //the 2nd primary component
      _[1]=Y
      _[2]=X2+4Z
   [2]:                                  //the 2nd prime component
      _[1]=Y
      _[2]=X2+4Z

//J2 is not radical

> ideal J3=slocus(radical(J2));
//J3 has 22 generators
> size(J3);
22
> ideal J4 = radical(J3);
J4[1]=Z
J4[2]=Y
J4[3]=X
```

`slocus(J)` return ideal of singular locus of J. In this case the radical of singular locus has two 1-dimensional components, one $L[1][2] := \langle 9Y^2+32XZ, X^2-12Z\rangle$ defines the cuspidal edge (blue color) and the other $L[2][2] := \langle Y, X^2+4Z\rangle$ defines the curve of double points (red color). The ideal $J4$ is the radical of ideal `slocus(radical(J2))` is the maximal ideal of ring R, and $V(J4) = \{(0,0,0)\}$ (green color) is the singular locus of `radical(J2)`. By Hilbert Nullstellensatz, $V(I) = V(\sqrt{I})$. Thus, the set of singular points in the Figure 2 was obtained by the vanishing locus of these radical ideals.

Example 2.10. Branches of space curve singularities

Consider the ideal $i = \langle x^4 - yz^2, xy - z^3, y^2 - x^3z\rangle$. As `dim(std(i))=1` we have a space curve, (`dim(std(i))` return the Krull dimension of i) and as `vdim(T_1(i))=13`, (`vdim(T_1(i))` return the tjurina number of i), this ideal defines a space curve in $\mathbb{C}[x, y, z]$ with isolated singularities. The objetive in this example, is to compute the number of branches and will be computed as an example of the pitfalls appearing in the use of primary decomposition. We can have two situations in which the primary decomposition algorithm might not lead to a complete decomposition. One of the computed components could be globally irreducible, but analytically reducible or could be irreducible over the rational numbers, but reducible over the field of complex numbers.

```
> ring r=0,(x,y,z),ds;
> ideal i=x^4-y*z^2,x*y-z^3,y^2-x^3*z;
> i;
i[1]=-yz2+x4
i[2]=xy-z3
i[3]=y2-x3z
> qhweight(i);
```

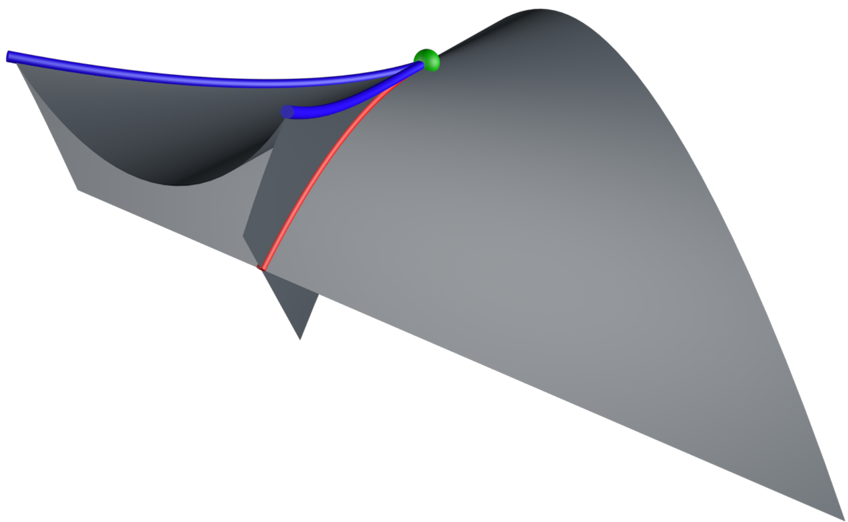


Figure 2: Swallowtail surface.

```
1,2,1
> dim(std(i));
1
```

As this space curve singularity is quasihomogeneous, we can pass to the polynomial ring. The command `qhweight(I)` computes the weight of the variables for a quasihomogeneous ideal I.

```
> ring r2=0,(x,y,z),dp;
ideal i=imap(r,i);
i[1]=x4-yz2
i[2]=-z3+xy
i[3]=-x3z+y2
> primdecGTZ(i);
[1]:
   [1]:
      _[1]=y-z2
      _[2]=x-z
   [2]:
      _[1]=y-z2
      _[2]=x-z
[2]:
   [1]:
      _[1]=y4+y3z2+y2z4+yz6+z8
      _[2]=y3+y2z2+yz4+xz5+z6
```

```
      _[3]=xy-z3
      _[4]=y2+xyz+yz2+x2z2+xz3
      _[5]=xy+yz+x3+x2z+xz2
   [2]:
      _[1]=y4+y3z2+y2z4+yz6+z8
      _[2]=y3+y2z2+yz4+xz5+z6
      _[3]=xy-z3
      _[4]=y2+xyz+yz2+x2z2+xz3
      _[5]=xy+yz+x3+x2z+xz2
> primdecSY(i);
[1]:
   [1]:
      _[1]=x-z
      _[2]=-y+z2
   [2]:
      _[1]=x-z
      _[2]=-y+z2
[2]:
   [1]:
      _[1]=-xy+z3
      _[2]=xy+yz+x3+x2z+xz2
      _[3]=y2+x2y+xyz+yz2+x2z2
   [2]:
      _[1]=-xy+z3
      _[2]=xy+yz+x3+x2z+xz2
      _[3]=y2+x2y+xyz+yz2+x2z2
```

Note that using the two algorithms for primary decompositions, the curve seems to have two branches. We can use other invariants of space curve singularities to check if this curve have two branches. Two known invariants are Milnor and Tjurina number. As this singularity curve is quasihomogeneous, we have the following formulae $\mu = \tau - t + 1$ and $\mu = 2\delta - r + 1$ where, τ is the Tjurina number, t is the Cohen-Macaulay type, r is the number of branches and δ is the delta invariant. As $\tau = 13$, $t = 2$, by the first formulae $\mu = 12$. On the other hand, if $r = 2$, by the second formulae μ is odd, but it is impossible. So obviously, we did not decompose this curve completely.

To compute the Cohen-Macaulay type, use minimal resolution `mres` or `CMtype` command.

```
> vdim(T_1(i));  \\ tjurina number
// dim T_1 = 13
13
> CMtype(i);  \\Cohen-Macaulay type
2
> resolution i_res = mres(i,0);    \\ minimal free resolution of an ideal
> i_res;
 1      3      2
```

```
r2 <--  r2 <--  r2

0       1       2
```

By the last number in the resolution, we see that the Cohen-Macaulay type of the given singularity is 2.

Let us now to compute the absolute prime components of i, using the command `absPrimdecGTZ(i)`, a procedure from library primdec.lib. We must assume ground field has characteristic 0. `absPrimdecGTZ(i)` return the list absolute primes such that each entry describes a class of conjugated absolute primes. The entry `absolute_primes[i][1]` return the absolute prime component, and `absolute_primes[i][2]` return the number of conjugates. We observe that the first entry of `absolute_primes[i][1]` is the minimal polynomial of a minimal finite field extension over which the absolute prime component is defined.

```
> def A=absPrimdecGTZ(i);

// 'absPrimdecGTZ' created a ring, in which two lists absolute_primes (the
// absolute prime components) and primary_decomp (the primary and prime
// components over the current basering) are stored.
// To access the list of absolute prime components, type (if the name S was
// assigned to the return value):
        setring S; absolute_primes;
> setring A;
> absolute_primes[1];
[1]:
   _[1]=a
   _[2]=x-z
   _[3]=z2-y
[2]:
   1
> absolute_primes[2];
[1]:
   _[1]=a4+a3+a2+a+1
   _[2]=z2+ya3+ya2+ya+y
   _[3]=xy+yza3+yza2+yza+yz
   _[4]=x3+x2z+xz2+xy+yz
[2]:
   4
```

Note that `absolute_primes[1]` has one component. The first entry of the list `absolute_primes[2][1]` is the polynomial $a^4+a^3+a^2+a+1$ of degree 4 in a which factors completely in 4 polynomials of type $a+b$ and `absolute_primes[2][2]=4` say that the second component of i has 4 branches. Finally, the ideal i has 5 branches.

$i = i_1 \cap i_2 \cap i_3 \cap i_4 \cap i_5$, where $I = \sqrt{-1}$ and

$$
\begin{aligned}
i_1 &= \langle x - z, z^2 - y\rangle,\\
i_2 &= \langle -\tfrac{1}{4}\, y\sqrt{5} + \tfrac{y}{4} + \tfrac{I}{4} y\sqrt{2}\sqrt{5+\sqrt{5}} + z^2, -\tfrac{1}{4}\, yz\sqrt{5} + \tfrac{1}{4}\, yz + \tfrac{I}{4} yz\sqrt{2}\sqrt{5+\sqrt{5}} + xy,\\
&\quad x^3 + x^2 z + xz^2 + xy + yz\rangle,\\
i_3 &= \langle \tfrac{1}{4}\, y\sqrt{5} + \tfrac{y}{4} + \tfrac{I}{4} y\sqrt{2}\sqrt{5-\sqrt{5}} + z^2, \tfrac{1}{4}\, yz\sqrt{5} + \tfrac{1}{4}\, yz + \tfrac{I}{4} yz\sqrt{2}\sqrt{5-\sqrt{5}} + xy,\\
&\quad x^3 + x^2 z + xz^2 + xy + yz\rangle,\\
i_4 &= \langle \tfrac{1}{4}\, y\sqrt{5} + \tfrac{y}{4} - \tfrac{I}{4} y\sqrt{2}\sqrt{5-\sqrt{5}} + z^2, \tfrac{1}{4}\, yz\sqrt{5} + \tfrac{1}{4}\, yz - \tfrac{I}{4} yz\sqrt{2}\sqrt{5-\sqrt{5}} + xy,\\
&\quad x^3 + x^2 z + xz^2 + xy + yz\rangle,\\
i_5 &= \langle -\tfrac{1}{4}\, y\sqrt{5} + \tfrac{y}{4} - \tfrac{I}{4} y\sqrt{2}\sqrt{5+\sqrt{5}} + z^2, -\tfrac{1}{4}\, yz\sqrt{5} + \tfrac{1}{4}\, yz - \tfrac{I}{4} yz\sqrt{2}\sqrt{5+\sqrt{5}} + xy,\\
&\quad x^3 + x^2 z + xz^2 + xy + yz\rangle.
\end{aligned}
$$

3 Multiple point spaces in the source

The multiple point spaces of a map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$ play an important role in the study of its geometry, as well as the topology of the discriminant of a stable perturbation. First, we present an explicit description of all stable types in the source and in the target for corank 1 map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$. This description is done in terms of subschemes of multiple points of a germ f. The description of the 0-stable types is shown by [Marar, Montaldi, Ruas, [26]]. In the following, the generalization for this description for all r-stable types, with $0 \leq r \leq n-1$, is shown by [Jorge-Pérez, Levcovitz, Saia,[15]]. In this chapter, our aim is to compute the stable types of f using a CA-system, such as SINGULAR. We present an implementation in Maple, [45], and SINGULAR and several examples will be calculated.

3.1 Notation and preliminaries

We follow the notation used by Gaffney in [7] and denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$.

For a germ $f \in \mathcal{O}(n,p)$, $J(f)$ denotes the ideal generated by the set of $p \times p$ minors of the derivative of f. The critical set of f, denoted by $\Sigma(f)$, is the set of points $x \in \mathbb{C}^n$ such that $J(f)(x) = 0$. The discriminant of f, denoted by $\Delta(f)$, is the image of the critical set by f.

Our interest is in $\mathcal{A}$-finitely determined map germs, where $\mathcal{A}$ denotes the usual Mather group of germs of holomorphic difeomorphisms in the source and in the target. We denote by F a versal unfolding of such a f.

Definition 3.1. We say that a stable type $\mathcal{Q}$ appears in F if for any representative $F = (id, f_u(x))$ of F, there exists a point $(s,y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$ is a stable germ of type $\mathcal{Q}$ where $S = f^{-1}(y) \cap \Sigma(f_u)$. The points (s,y) and (s,x) with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.

Definition 3.2. We say that $\mathcal{Q}$ is a zero-dimensional stable type for the pair (n,p) if $\mathcal{Q}(f)$ has dimension 0 where f is a representative of the stable type $\mathcal{Q}$.

A finitely determined map germ f has discrete stable type if there exist a versal unfolding F of f in which appears only a finite number of stable types. If (n,p) is in the nice range of dimensions or in this boundary, then any finitely determined germ f has a discrete stable type.

We say that two elements f and g of $\mathcal{O}(n,p)$ are $\mathcal{A}$-*equivalent* if $f = l \circ g \circ r^{-1}$ where $(l,r) \in \mathrm{Diff}(\mathbb{C}^n, 0) \times \mathrm{Diff}(\mathbb{C}^p, 0)$.

A germ is k-$\mathcal{A}$-*determined* if any $g \in \mathcal{O}(n,p)$ with the same k-jet as f, i.e. $j^k g = j^k f$, is $\mathcal{A}$-equivalent to f. The germ f is said to be $\mathcal{A}$-*finitely determined or finitely determined* if it is k-$\mathcal{A}$-determined for some k.

Mather and Gaffney gave the characterization of finitely determined map germs in terms of stable germs, [44]. We say f is a *stable germ* if every nearby germ is $\mathcal{A}$-equivalent to f.

A **deformation** to s-parameters of a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a map germ f_s em $\mathcal{O}(s+n, p)$ such that

$$
\begin{aligned}
f_s : (\mathbb{C}^s \times \mathbb{C}^n, (0,0)) &\to (\mathbb{C}^p, 0) \\
(u, x) &\mapsto f_s(u, x)
\end{aligned}
$$

where, $f_s(0, x) = f(x)$.

An **unfolding** to s-parameters of f is a map germ in $\mathcal{O}(s+n, s+p)$,

$$
\begin{gathered}
F : (\mathbb{C}^s \times \mathbb{C}^n, (0,0)) \to (\mathbb{C}^s \times \mathbb{C}^p, (0,0)) \\
(u, x) \mapsto F(u, x) = (u, f_s(u, x))
\end{gathered}
$$

where f_s is a deformation of f.

F is a **trivial unfolding** of f if there are unfoldings to s-parameters, R of identity in $\mathbb{C}^n$ and L, of identity in $\mathbb{C}^p$ such that $L \circ F \circ R^{-1} = (id, f)$.

Two unfoldings F_1 and F_2 of f are **isomorphic** if $F_2 = L \circ F_1 \circ R^{-1}$ where L and R are unfoldings of identity.

A s-parameter unfolding F_2 is **induced** by an unfolding $F_1 = (id, f_s)$ to t-parameters, if exists a map $h : (\mathbb{C}^s, 0) \to (\mathbb{C}^t, 0)$ such that F_2 is equal to $h^* F_1 = (id, f_s(h(s), x))$.

F is a **versal unfolding** of f if all unfolding of f is isomorphic to an induced unfolding of F.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finitely determined map germ . We say f is **stable** if any unfolding F of it is trivial.

Definition 3.3. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finitely determined map germ. A 1-parameter* unfolding *of f is a map germ $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^p, 0)$ of the form $F(t, x) = (t, f_t(x))$ such that $f_0 = f$. We say that an unfolding F is a* stabilization *of f if there is a representative $F : D \times U \to D \times \mathbb{C}^p$, where D, U are open neighbourhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively such that $f_t : U \to \mathbb{C}^p$ is stable for any $t \in D \setminus \{0\}$.*

In range of nice dimensions in sense of Mather, it is known that a stabilization of a finitely determined map germ always exist.

Next theorem relates finite determinacy with stability.

Theorem 3.4. *[27] Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map germ $\mathcal{K}$-finitely determined. Then, this germ is $\mathcal{A}$-finitely determined if, and only if, for each representative of f, there exists a neighborhood U of 0 in $\mathbb{C}^n$ and V of 0 in $\mathbb{C}^p$, $f(U) \subset V$, such that if $y \in V - \{0\}$, $f^{-1}(y) \cap \Sigma(f) \cap U = \{x_1, \ldots, x_r\}$, then the multi-germ of f in $\{x_1, \ldots, x_r\}$ is $\mathcal{A}$-stable.*

3.2 The stable types in $\mathcal{O}(n,n)$

As highlighted in the introduction, our aim is to compute the stable types of f using a CA-system.

First we present the mathematical methods to obtain these stable types in the source. We present here an explicit description of all stable types in the source and in the target for corank 1 map germs from $(\mathbb{C}^n,0)$ to $(\mathbb{C}^n,0)$. This description is done in terms of subschemes of multiple points of a germ f. The description of the 0-stable types is shown by [Marar, Montaldi, Ruas, [26]]. In the following, the generalization for this description for all r-stable types, with $0 \le r \le n-1$, is shown by [Jorge-Pérez, Levcovitz, Saia,[15]].

For this, we first give the following preliminary definition. Given a continuous mapping $f : X \to Y$ on analytic spaces, we define the k^{th} multiple points space of f as

$$D^k(f) := \text{closure}\left\{(x_1, x_2, ..., x_k) \in X^k : f(x_1) = ... = f(x_k) \ \text{ for } x_i \neq x_j, i \neq j\right\}.$$

Let $f \in \mathcal{O}(n,n)$ be a finitely determined map germ of corank 1. Choosing linearly adapted coordinates, we can write $f(x,z) = (x_1,\ldots,x_{n-1},g(x,z))$, where $x = (x_1,\ldots,x_{n-1}) \in \mathbb{C}^{n-1}$, $z \in \mathbb{C}$ and $g : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ is a polynomial. For each partition $\mathcal{P} = (r_1,\cdots,r_\ell)$ of an integer $m \le n$, i.e, $r_1+\cdots+r_\ell = m$, we consider the subset $D^\ell(f,\mathcal{P})$ of $\mathbb{C}^{n-1}\times\mathbb{C}^\ell$, with $\ell := \text{length}(\mathcal{P})$, defined by

$$D^\ell(f,\mathcal{P}) = closure\left\{ (x,z_1,\cdots,z_\ell)\in\mathbb{C}^{n-1}\times\mathbb{C}^\ell : \begin{array}{l} z_i \neq z_j, \\ f(x,z_i) = f(x,z_j) \text{ and} \\ f \ \text{ has a singularity of type} \\ A_{r_j} \text{ at } (x,z_j) \end{array} \right\}$$

where ‘closure’ means the analytic closure in $\mathbb{C}^{n-1}\times\mathbb{C}^\ell$.

According to [26], if a corank 1 map germ $f : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ is $\mathcal{A}$-finitely determined, then the singularity of f at 0 splits up into a number of non-degenerate *zero*-dimensional stable singularities of a stable perturbation of f. A stable map germ $f : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0))$ has an A_k singularity ($k \le n$) if it is $\mathcal{A}$-equivalent to the germ, $G(x_1,\ldots,x_{n-1},z) = (x_1,\ldots,x_{n-1},z^{k+1}+x_1z^{k-1}+\cdots+x_{k-1}z)$. Moreover, any stable corank 1 map germ is an A_k for some natural number k.

We remark that when $m = n$, the subsets $D^n(f,\mathcal{P})$ are called zero-schemes and are related to the 0-stable types [26]. We will soon give a structure of subschemes to the sets $D^\ell(f,\mathcal{P})$ as well. Nearby the $(A_{r_1}+\cdots+A_{r_\ell}) = A_{r_1,\cdots,r_\ell}$ multi-germs, there are points in the target with $(r_1+1)+(r_2+1)+\cdots+(r_\ell+1) = m+\ell$ pre-images. We define a $(m+\ell)-$tuple scheme in $\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}$, which, on the appropriate diagonal, specializes to the ideal defining $A_{r_1,\cdots,r_\ell}$ multi-germs. (See Lemma 3.5).

We denote the coordinates of $\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}$ by

$$(x,\mathbf{z}) = (x,z_0^1,\cdots,z_{r_1}^1,z_0^2,\cdots,z_{r_2}^2,\cdots,z_0^\ell,\cdots,z_{r_\ell}^\ell)$$

and define the sheaf of ideals $\mathcal{J}^\ell(f,\mathcal{P}) = \langle h_1,h_2,\cdots,h_{m+\ell-1}\rangle \subset \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}}$ by

$$h_i := \frac{\begin{vmatrix} 1 & z_0^1 & \cdots & (z_0^1)^{i-1} & g_0^1 & (z_0^1)^{i+1} & \cdots & (z_0^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_1}^1 & \cdots & (z_{r_1}^1)^{i-1} & g_{r_1}^1 & (z_{r_1}^1)^{i+1} & \cdots & (z_{r_1}^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_0^\ell & \cdots & (z_0^\ell)^{i-1} & g_0^\ell & (z_0^\ell)^{i+1} & \cdots & (z_0^\ell)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_\ell}^\ell & \cdots & (z_{r_\ell}^\ell)^{i-1} & g_{r_\ell}^\ell & (z_{r_\ell}^\ell)^{i+1} & \cdots & (z_{r_\ell}^\ell)^{m+\ell-1} \end{vmatrix}}{\begin{vmatrix} 1 & z_0^1 & \cdots & (z_0^1)^{i-1} & (z_0^1)^i & (z_0^1)^{i+1} & \cdots & (z_0^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_1}^1 & \cdots & (z_{r_1}^1)^{i-1} & (z_{r_1}^1)^i & (z_{r_1}^1)^{i+1} & \cdots & (z_{r_1}^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_0^\ell & \cdots & (z_0^\ell)^{i-1} & (z_0^\ell)^i & (z_0^\ell)^{i+1} & \cdots & (z_0^\ell)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_\ell}^\ell & \cdots & (z_{r_\ell}^\ell)^{i-1} & (z_{r_\ell}^\ell)^i & (z_{r_\ell}^\ell)^{i+1} & \cdots & (z_{r_\ell}^\ell)^{m+\ell-1} \end{vmatrix}} \tag{3.1}$$

note that the denominator above is the Vandermonde determinant of the list $(z_0^1, \cdots, z_{r_1}^1, \cdots, z_0^\ell, \cdots, z_{r_\ell}^\ell$ and $g_i^k := g(x, z_i^k)$. In $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$ there is a diagonal of particular interest, namely,

$$\Delta(\mathcal{P}) = \{(x, \mathbf{z}) \in \mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell} | z_i^k = z_j^k,\ \forall i, j = 0, \cdots, r_k,\ \forall k = 1, \cdots, \ell\}$$

which can be parametrized by $(x, z^1, \cdots, z^\ell)$:

$$(x, \mathbf{z}) = (x, z^1, \cdots, z^1, z^2, \cdots, z^2, \cdots, z^\ell, \cdots, z^\ell) \tag{3.1}$$

with z^i repeated $r_i + 1$ times. This corresponds to an embedding $j_\ell : \mathbb{C}^{n-1} \times \mathbb{C}^\ell \to \mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$. We denote by $j_\ell^* : \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}} \to \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}$ the induced surjection.

Let $\mathcal{I}_{\Delta(\mathcal{P})} = \langle z_i^k - z_0^k, i = 1, \cdots, r_k; k = 1, \cdots, \ell\rangle$ be the ideal defining $\Delta(\mathcal{P})$. Then j_ℓ^* induces an isomorphism $\frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}}}{\mathcal{I}_{\Delta(\mathcal{P})}} \cong \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}$. Note that a generic point of $V(\mathcal{I}_{\Delta(\mathcal{P})})$ is one of the form (3.1) with $z^i \neq z^j$, for $i \neq j$.

Let $\mathcal{J}_\Delta^\ell(f, \mathcal{P})$ be the sheaf of ideals in $\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}}$ defined by

$$\mathcal{J}_\Delta^\ell(f, \mathcal{P}) := \mathcal{J}^\ell(f, \mathcal{P}) + \mathcal{I}_{\Delta(\mathcal{P})}$$

and consider the ideal $\mathcal{I}^\ell(f, \mathcal{P}) := j_\ell^*(\mathcal{J}_\Delta^\ell(f, \mathcal{P}))$ of $\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}$. Then j_ℓ^* also induces an isomorphism

$$j_\ell^* : \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^{m+\ell}}}{\mathcal{J}_\Delta^\ell(f, \mathcal{P})} \to \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}}{\mathcal{I}^\ell(f, \mathcal{P})}. \tag{3.2}$$

Next lemma shows us that at a generic point of $\Delta(\mathcal{P})$ we have $D^\ell(f, \mathcal{P}) = V(\mathcal{J}_\Delta^\ell(f, \mathcal{P}))$, that is, f has a singularity of type A_{r_j} at (x, z^j) and $f(x, z^1) = \cdots = f(x, z^\ell)$. Therefore, by an abuse of notation, we will call the subscheme $V(\mathcal{I}^\ell(f, \mathcal{P})) \subset \mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}$ equipped with the structural sheaf $\frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}}{\mathcal{I}^\ell(f,\mathcal{P})}$ by $D^\ell(f, \mathcal{P})$ as well.

Lemma 3.5. *([24], lemma 2.7) At a generic point of $\Delta(\mathcal{P})$ we have,*

$$\mathcal{J}^{\ell}_{\Delta}(f,\mathcal{P}) \;=\; \left\langle \tfrac{\partial g}{\partial z}(x,z^1),\cdots,\tfrac{\partial^{r_1} g}{\partial z^{r_1}}(x,z^1),\cdots,\tfrac{\partial g}{\partial z}(x,z^{\ell}),\cdots,\tfrac{\partial^{r_\ell} g}{\partial z^{r_\ell}}(x,z^{\ell})\right\rangle$$

$$+\langle g(x,z^i)-g(x,z^1);\ 2\leq i\leq \ell\rangle+\mathcal{I}_{\Delta(\mathcal{P})}.$$

We remark that for the partition $\mathcal{P}=(1,\cdots,1)$ of m with $1+\cdots+1=m=\ell$, we have $D^{\ell}(f,\mathcal{P})=D^{\ell}(f)$, where $D^{\ell}(f)$ denotes the set of ordinary ℓ-multiple points of f.

For the partition $\mathcal{P}=(r_i)$, then $r_i=m$, $\ell=1$, and $D^1(f,\mathcal{P})=\Sigma^{1,\cdots,1}(f)$, where $\Sigma^{1,\cdots,1}(f)$ is the set of singularities of type $\Sigma^{1,\dots,1}$ of f with $1,\cdots,1$ repeated r_i-times, i.e, f has a singularity of type A_{r_i}.

Proposition 3.6 ([15])**.** *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ be a finitely determined map germ of corank 1. Then, for any partition $\mathcal{P}=(r_1,\dots,r_\ell)$ of $m\leq n$ we have:*

1. *f is stable if and only if $V\left(\mathcal{J}^{\ell}_{\Delta}(f,\mathcal{P})\right)$ is smooth of dimension $n-m$, or empty;*
2. *The ideal $\mathcal{J}^{\ell}_{\Delta}(f,\mathcal{P})$ at 0 is either an ICIS of dimension $n-m$, or is empty;*

Proof We denote by H^{ℓ} the map germ defined by the generators h_i of the ideal $\mathcal{J}^{\ell}(f,\mathcal{P})$ and $E(\mathcal{P})$ the map germ defined by the generators of the ideal $\mathcal{I}_{\Delta,\mathcal{P}}$.

To prove the item 1., we have from 2.13 of [24] that f is stable of type A_{r_i} at 0 if and only if the map germ $\left(H^{\ell},E(\mathcal{P})\right)$ is a submersion, and this is equivalent to say that $D^{\ell}(f,\mathcal{P})$ is smooth of dimension $n-m$.

2. Suppose that f is finitely determined and choose a representative $f:U\to V$ as in the Geometric Theorem of Mather-Gaffney (see [44]), we shall show that for any partition $\mathcal{P}$ of an $m\leq n$ and at any point $(x,\mathbf{z})\neq(0,0)$ the mapping $(H^{\ell},E(\mathcal{P}))$ is a submersion. If necessary we restrict U such that f is a singularity of type $A_{r_1,\dots,r_\ell}$, after reordering if necessary, we can suppose that $(x,\mathbf{z})$ is a generic point of $\mathcal{I}_{\Delta(\mathcal{P})}$ for some partition $\mathcal{P}=(r_1,\dots,r_\ell)$, hence $(x,\mathbf{z})=(x,z^1,\cdots,z^1,z^2,\cdots,z^2,\cdots,z^{\ell},\cdots,z^{\ell})$. Now we suppose that f is a singularity of type A_{r_i} in (x,z^i). It follows by the Geometric Theorem that the multi-germ of f at $\left\{(x,z^1),\dots,(x,z^{\ell})\right\}$ is stable, then by the theorem 2.13 of [24] the mapping $(H^{\ell},E(\mathcal{P}))$ is a submersion, thus for any point distinct from 0, the $2m+\ell-1$ functions generating $\mathcal{J}^{\ell}_{\Delta}(f,\mathcal{P})$ define a submersion, therefore the variety $V\left(\mathcal{J}^{\ell}_{\Delta}(f,\mathcal{P})\right)$ is an ICIS at 0 of dimension $n-m$. □

We will give now an explicit description of the stable types in the source and the target of any finitely determined map germ $f\in\mathcal{O}(n,n)$ with corank 1.

For each partition $\mathcal{P}=(r_1,\dots,r_\ell)$ of $m\leq n$, we denote by $D^{\ell}_1(F,\mathcal{P})$ the projection of $D^{\ell}(F,\mathcal{P})$ to the (x,z)-space. We remember that each of the sets $D^{\ell}_1(F,\mathcal{P})$ is a subset of $\Sigma(f)$.

In the next theorem we will use the following notation.

Let $X(f):=\overline{(f^{-1}(\Delta(f))-\Sigma(f))}$. For each partition $\mathcal{P}$ of $m\leq n$, we define the sets $X^{\ell}_1(f,\mathcal{P})$ by,

$$X^{\ell}_1(f,\mathcal{P}):=f^{-1}(f(D^{\ell}_1(f,\mathcal{P})))-\left(D^{\ell}_1(f,\mathcal{P})\cap\Sigma(f)\right).$$

Theorem 3.7 ([15]). *Let $f \in \mathcal{O}(n,n)$ be a finitely determined map germ of corank 1. Then,*

1. *The stables types in the* **source** *are $D_1^\ell(f,\mathcal{P}) \subset \Sigma(f)$ and $X_1^\ell(f,\mathcal{P}) \subset X(f)$, for all partitions $\mathcal{P} = (r_1,\ldots,r_\ell)$ of all $m \leq n$.*

2. *The stables types in the* **target** *are $f(D_1^\ell(f,\mathcal{P})) \subset \Delta(f)$, for all partitions $\mathcal{P} = (r_1,\ldots,r_\ell)$ of each $m \leq n$.*

3. *The dimensions of $X_1^\ell(f,\mathcal{P})$ and of $f(D_1^\ell(f,\mathcal{P}))$ are both $n-m$.*

Proof 1. A stable map germ $f \in \mathcal{O}(n,n)$ has an A_k singularity if it is $\mathcal{A}$-equivalent to the germ

$$(x_1, ..., x_{n-1}, z) \to (x_1, ..., x_{n-1}, z^{k+1} + x_1 z^{n-1} + ... + x_{k-1}z).$$

Moreover, any stable corank 1 map germ is an A_k singularity for some natural number k, hence the set of points in $\mathbb{C}^n$ where a stable map has an A_k singularity is a smooth sub-manifold of codimension k. The image of this set by f is also a smooth sub-manifold of codimension k. Since f is finitely determined, it follows by the Geometric criterion of Mather-Gaffney, that there exist neighborhoods U and V of 0 in $\mathbb{C}^n$ such that $f^{-1}(0) \cap U \cap \Sigma(f) = 0$ and for each $y \in V$, $y \neq 0$, the germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^n, y)$ is stable ($S = f^{-1}(y) \cap U \cap \Sigma(f)$), hence for each $x \in S$, the germ $f : (\mathbb{C}^n, x) \to (\mathbb{C}^n, y)$ is an A_k for some k and these sub-manifolds in the discriminant are in general position. But this occurs if and only if $r_1+r_2+.....+r_j = m \leq n$. We call such multi germ and $A_{\mathcal{P}}$-singularity and the result follows from the Lemma 3.5.

2. and 3. From the corollary given in the page 19 of [12], we know that there exist neighborhoods of the origin U_1 in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$ and U_2 in $\mathbb{C}^n$ such that the map $p : D^m(f,\mathcal{P}) \to U_2$ induced by the projection $U_1 \to U_2$ is proper and finite. Since f is proper and finite, the map $f \circ p$ is also proper and finite, then $V = (f \circ p)(D^m(f,\mathcal{P}))$ is an analytic subvariety $n-m$-dimensional, in particular, $f(D_1^\ell(f,\mathcal{P}))$ is $n-m$-dimensional. Since $D_1^\ell(f,\mathcal{P})$ is $n-m$-dimensional and f is proper and finite, we also obtain that $X_1^\ell(f,\mathcal{P})$ is an analytic space of dimension $n-m$. □

According to Proposition 3.7, (see too [26]), given an $\mathcal{A}$-finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, and a partition $\mathcal{P}$ of n, how many $A_{\mathcal{P}}$ singularities are there in a stabilization of f, in a suitably small neighbourhood of 0? This number is independent of the particular stabilization choosen, and we denote it $\sharp A_{\mathcal{P}}(f)$.

Theorem 3.8 ([26], Theorem 1). *Let $f(x_1,\ldots,x_{n-1},z) = (x_1,\ldots,x_{n-1},g(x,z))$ be $\mathcal{A}$-finitely determined weighted-homogeneuos map germ, with weights $(\omega_1,\ldots,\omega_{n-1},\omega_0)$ and g with weighted-degree d. For any stabilization of f and any partition $\mathcal{P}$ of n*

$$\#A_p = \frac{\omega_0^{n-1}}{N(\mathcal{P})\omega} \prod_{j=1}^{n+\ell-1} (\frac{d}{\omega_0} - j)$$

with $\omega = \prod_{i=1}^{n-1} \omega_i$.

If the map germ f is of corank 1 and $\mathcal{A}$-finite, but not weighted-homogeneuos, then the $\sharp A_{\mathcal{P}}(f)$, can be computed by the following result.

Theorem 3.9 ([26], Corollary 5)**.** *Let $f(x_1, \ldots, x_{n-1}, z) = (x_1, \ldots, x_{n-1}, g(x, z))$ be $\mathcal{A}$-finitely determined map germ. For any stabilization of f and any partition $\mathcal{P}$ of n*

$$\#A_p = \frac{1}{N(\mathcal{P})} \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n-1}\times\mathbb{C}^\ell}}{\mathcal{I}^\ell(f, \mathcal{P}))}.$$

$N(\mathcal{P})$ is the order of the subgroup of the permutation group S_ℓ which fixes $\mathcal{P}$. S_ℓ acts on $\mathbb{C}^\ell$ by permuting the coordinates.

3.3 Algorithm for calculating ideals $\mathcal{I}^\ell(f, \mathcal{P})$

We show here a generalization, for all $m \leq n$, of the algorithm given in [26] for the case $m = n$. This algorithm IDEALSOURCE computes the ideals $\mathcal{I}^\ell(f, \mathcal{P}))$ of the stable types in the source for all partition $\mathcal{P}$ of $m \leq n$. We implemented this algorithm using the software Maple and SINGULAR. See Subsection 5.1 for source code in SINGULAR and Maple.

IDEALSOURCE

INPUTS: $f := (x, g(x, z))$ and $\mathcal{P} = (r_1, \ldots, r_\ell)$ of $m \leq n$.
Create variables $L := (x, \mathbf{z}) = (x, z_0^1, \cdots, z_{r_1}^1, z_0^2, \cdots, z_{r_2}^2, \cdots, z_0^\ell, \cdots, z_{r_\ell}^\ell)$.
FOR $n = 1, \ldots, m + \ell - 1$ do:
V:= Vandermonde(L);
A:=V;
FOR $i = 1, \ldots, m + \ell - 1$ do:
A:=subs(z = A[i,2], g);
End i;
H[n-1]:=subs(Diagonal, det(A)/det(V));
End n;
OUTPUT: < H[1] ,..., H[m+ℓ-1] >.

IDEALSOURCE.LIB is a library in SINGULAR available in [28].

Remark 3.10. IDEALSOURCE works with a ring of parameters, but with unfolding can be faster. For example if $f(x, y) = (x, y^4 + xy)$ and $f_u(x, y) = (x, y^4 + xy + uy^2)$ and partition = [1,1], use:

With parameters: `ring r=(0,u),(x,y),dp;idealsource([x,y4+xy+uy2],[1,1]);`

As unfolding: `ring r=0,(u,x,y),ds;idealsource([u,x,y4+xy+uy2],[1,1]);`

Below applications about good real and maximal deformation by using IDEALSOURCE.

Definition 3.11. *A stable deformation of a finitely determined real map germ from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^n, 0)$ is* **maximal** *(or an M-deformation) if it exhibits all of the 0-dimensional stable singularities present in its complexification.*

Theorem 3.12 (Rieger-Ruas, [39]). *All $\mathcal{A}$-simple rank $n-1$ germs $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, where $m \geq n$, have an M-deformation.*

Definition 3.13. *A stable deformation of a finitely determined real map germ from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^n, 0)$ is a* **good real deformation** *if the real image has $(n-1)$th homology of rank $\mu_\Delta(f)$ (discriminant Milnor number), so that inclusion of real image in complex image induces an isomorphism on H_n.*

Theorem 3.14 ([4], Theorem 4.6). *The discriminant $\Delta(f_t)$ of a stabilisation or weak stabilisation of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \geq p$, has the homotopy type of a wedge of spheres of dimension $p-1$.*

3.4 Application 1: Good real deformation for $f(x, z) = (x, z^6 + xz)$

For finitely determined map germs $f \in \mathcal{O}(2,2)$ of corank 1, $D^2(f,(1,1)) = D^2(f) \subset \mathbb{C}^2 \times \mathbb{C}$ is the set of source double points of f and $D_1^2(f,(1,1)) = D(f) \subset \Sigma(f) \subset \mathbb{C}^2$. $D^1(f,(2)) = \Sigma^{1,1}(f) \subset \Sigma(f)$ is the set of cusps of f. We denote by $A_{1,1}(f) = f(D(f))$ and $A_2(f) = f(\Sigma^{1,1}(f))$ the 0-stable singularities on target. Then $A_{1,1}(f)$ is the set of target ordinary double points of f and $A_2(f)$ is the set of cusps of f in the target.

Normal form for stable types in $\mathcal{O}(2,2)$:

For finitely determined map germs $f \in \mathcal{O}(2,2)$, the stable types are in the tables Table 1.1 and Table 1.2 with its normal form.

Table 1.1: **Stable monogermes in $\mathcal{O}(2,2)$**

Name	$\Sigma^{(i)}$	A_k	*Normal form*
Submersion	$\Sigma^{(0)}$	A_0	(x, z)
Fold	$\Sigma^{(1)}$	A_1	(x, z^2)
Cusp	$\Sigma^{(1,1)}$	A_2	$(x, z^3 + xz)$

Table 1.2: **Stable multigerms in $\mathcal{O}(2,2)$**

Name	A_k	*Normal form*
Ordinary double points	$A_{1,1}$	$\{(x, z^2); (x^2, z)\}$

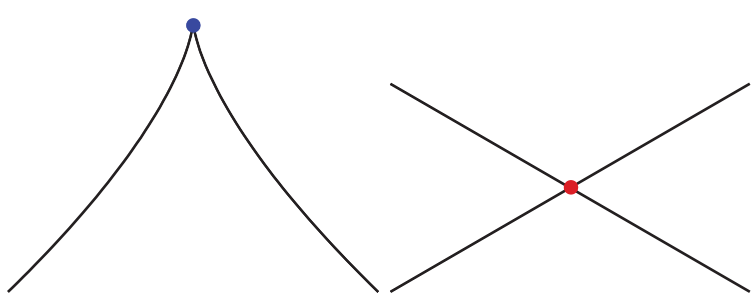


Figure 3: 0-stable types: $A_2 : (x, y^3 + xy)$ and $A_{1,1} : \{(x, y^2); (x^2, y)\}$.

The notation for A_k singularities was introduced by Arnol'd, [1]. The symbol Σ^i is the notation of Thom-Boardman, for singularities of type Σ^i, [2].

Theorem 3.15 ([4],Lemma 2.3). $\mu_\Delta(f) = \mu(\Sigma(f)) + \sharp A_{1,1} = \mu(\Delta(f)) - (2\sharp A_2 + \sharp A_{1,1})$.

Remark 3.16. $\mu_\Delta(f)$ is the discriminant Milnor number and $\mu(\Delta(f))$ is the usual Milnor number, here $\Delta(f)$ is a curve with isolated singularity.

Example 3.17. Consider the map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, defined by:

$$f(x, z) = (x, z^6 + xz).$$

Rieger studied the geometry of the real stable perturbations of this f, [38].
Some invariants of this germ are:

$$\sharp A_2 = 4, \quad \sharp A_{1,1} = 6, \quad \mu(\Sigma(f)) = 0, \quad \mu(\Delta(f)) = 20$$

and the discriminant set $\Delta(f_t)$ has the homotopy type of a wedge of 6 circles, since that

$$\mu_\Delta(f) = 6.$$

Consider de following versal deformation $f_{v,u,t,w,p,q}(x, z) = (x, z^6 + xz + vz^2 + uz^3 + tz^4 + wz^8 + pz^9 + qz^{14})$ of f. Using the algorithm `idealsource`, for the partitions [1,1] and [2] of $m = n = 2$, we obtain the following ideals:

```
> LIB "idealsource.lib";
> ring r=(0,v,u,t,w,p,q),(x,z),dp;
> idealsource([x,z^6+xz+vz2+uz3+tz4+wz8+pz9+qz14],[1,1]);
//f:(C^2,0)--->(C^2,0) ; Partition = [1,1]

H[1]=22*q*z01^12*z02+52*q*z01^11*z02^2+76*q*z01^10*z02^3+94*q*z01^9*z02^4
      +106*q*z01^8*z02^5+112*q*z01^7*z02^6+112*q*z01^6*z02^7+
      106*q*z01^5*z02^8+94*q*z01^4*z02^9+76*q*z01^3*z02^10+
      52*q*z01^2*z02^11+22*q*z01*z02^12+12*p*z01^7*z02+27*p*z01^6*z02^2+
      36*p*z01^5*z02^3+39*p*z01^4*z02^4+36*p*z01^3*z02^5+27*p*z01^2*z02^6+
      12*p*z01*z02^7+10*w*z01^6*z02+22*w*z01^5*z02^2+28*w*z01^4*z02^3+
      28*w*z01^3*z02^4+22*w*z01^2*z02^5+10*w*z01*z02^6+6*z01^4*z02+
      12*z01^3*z02^2+12*z01^2*z02^3+6*z01*z02^4+2*t*z01^2*z02+
      2*t*z01*z02^2+x
H[2]=11*q*z01^12+44*q*z01^11*z02+71*q*z01^10*z02^2+92*q*z01^9*z02^3+
      107*q*z01^8*z02^4+116*q*z01^7*z02^5+119*q*z01^6*z02^6+
      116*q*z01^5*z02^7+107*q*z01^4*z02^8+92*q*z01^3*z02^9+71*q*z01^2*z02^10
      +44*q*z01*z02^11+11*q*z02^12+6*p*z01^7+24*p*z01^6*z02+36*p*z01^5*z02^2
      +42*p*z01^4*z02^3+42*p*z01^3*z02^4+36*p*z01^2*z02^5+24*p*z01*z02^6+
      6*p*z02^7+5*w*z01^6+20*w*z01^5*z02+29*w*z01^4*z02^2+32*w*z01^3*z02^3+
      29*w*z01^2*z02^4+20*w*z01*z02^5+5*w*z02^6+3*z01^4+12*z01^3*z02+
      15*z01^2*z02^2+12*z01*z02^3+3*z02^4+t*z01^2+4*t*z01*z02+t*z02^2-v
H[3]=12*q*z01^11+22*q*z01^10*z02+30*q*z01^9*z02^2+36*q*z01^8*z02^3+
      40*q*z01^7*z02^4+42*q*z01^6*z02^5+42*q*z01^5*z02^6+40*q*z01^4*z02^7+
      36*q*z01^3*z02^8+30*q*z01^2*z02^9+22*q*z01*z02^10+12*q*z02^11+
```

```
       7*p*z01^6+12*p*z01^5*z02+15*p*z01^4*z02^2+16*p*z01^3*z02^3+
       15*p*z01^2*z02^4+12*p*z01*z02^5+7*p*z02^6+6*w*z01^5+10*w*z01^4*z02+
       12*w*z01^3*z02^2+12*w*z01^2*z02^3+10*w*z01*z02^4+6*w*z02^5+4*z01^3+
       6*z01^2*z02+6*z01*z02^2+4*z02^3+2*t*z01+2*t*z02+u
//To access the ideal H, type:  setring MR; H;

> ring r=(0,v,u,t,w,p,q),(x,z),dp;
> idealsource([x,z^6+xz+vz2+uz3+tz4+wz8+pz9+qz14],[2]);
//f:(C^2,0)--->(C^2,0) ; Partition = [2]
H[1]=168*q*z01^13+63*p*z01^8+48*w*z01^7+24*z01^5+8*t*z01^3+3*u*z01^2-x
H[2]=91*q*z01^12+36*p*z01^7+28*w*z01^6+15*z01^4+6*t*z01^2+3*u*z01+v
```

Analysing these ideals, there exists real parameters v, u, t, w, p, q such that all ten 0-stable singularities, appear in real coordinates. For instance if $u = w = p = q = 0$, $v = -t$ and $t < -\frac{5}{3}$, then all 4 cusps appear in real coordinates in the discriminant. If $u = w = p = q = 0$, $v = -t$ and $t \in (-\frac{60}{11}, -4) \cup (-4, -3)$, then all 6 distinct double points appear in real coordinates in the discriminant. See Figure 4 (left).

With $z01 = z, z02 = z_2$, $u = w = p = q = 0$, $v = -t$, we have the following 1-parameter deformation $f_t(x, z) = (x, z^6 + xz - tz^2 + tz^4)$, and

$$\begin{aligned} D^2(f_t,(1,1)) \;\; = \;\; & V(\langle 6z^4z2 + 12z^3z2^2 + 12z^2z2^3 + 6zz2^4 + 2tz^2z2 + 2tzz2^2 + x, \\ & -3z^4 - 12z^3z2 - 15z^2z2^2 - 12zz2^3 - 3z2^4 - tz^2 - 4tzz2 - tz2^2 - t, \\ & 4z^3 + 6z^2z2 + 6zz2^2 + 4z2^3 + 2tz + 2tz2\rangle). \end{aligned}$$

$D^1(f_t,(2)) = V(\langle -24z^5 - 8tz^3 + x, 15z^4 + 6tz^2 - t\rangle$.

$D^2(f_t,(1,1)) \subset \mathbb{C}(t)[x, z, z_2]$ is the 0-dimensional double points set of $f_t(x, y)$.

$D^1(f_t,(2)) \subset \mathbb{C}(t)[x, z]$ is the 0-dimensional cusp of $f_t(x, y)$.

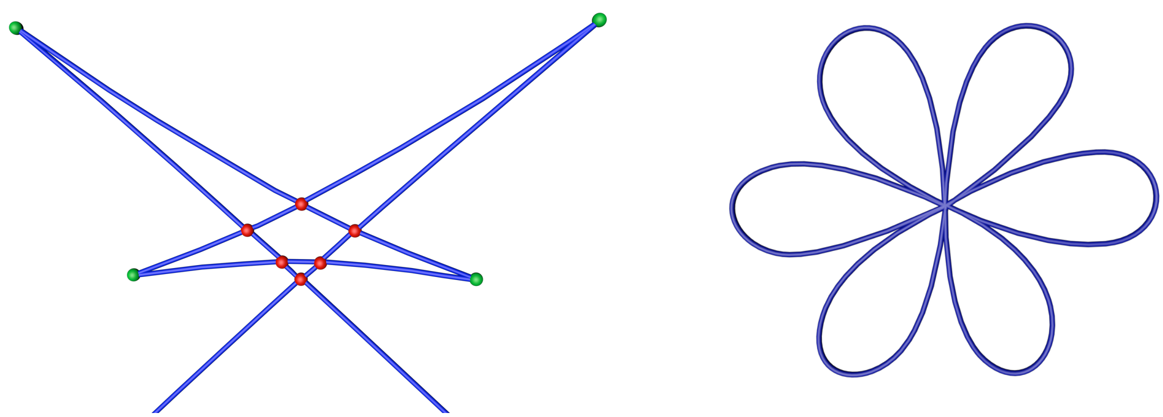


Figure 4: $\Delta(f_{u,v})$ and wedge of 6-circles.

Therefore, this germ is maximal because all ten 0-stable singularities appear with real coordinates and has a good real deformation because the number of circles in bouquet is $6 = \mu_\Delta(f)$, see Figure 4.

3.5 Application 2: Real geometry for the germ $f_{u,v}(x,y,z) = (x,y,z^6+yz+xz^2+uz^3+vz^4)$

The objective in this example is to calculate all possible real r-stable singularities of the following deformation $f_{u,v}(x,y,z) = (x,y,z^6+yz+xz^2+vz^4+uz^3)$ of the map germ $f(x,y,z) = (x,y,z^6+yz+xz^2)$.

First we show in the tables Table 1.1 and Table 1.2 the normal form for stable germs and multigerms in $\mathcal{O}(3,3)$.

Table 1.1: **Stable monogermes in $\mathcal{O}(3,3)$**

Name	$\Sigma^{(i)}$	A_k	*Normal form*
Submersion	$\Sigma^{(0)}$	A_0	(x,y,z)
Fold	$\Sigma^{(1)}$	A_1	(x,y,z^2)
Cuspidal edge	$\Sigma^{(1,1)}$	A_2	(x,y,z^3+xz)
Swallowtail	$\Sigma^{(1,1,1)}$	A_3	$(x,y,z^4+xz+yz^2)$

Table 1.2: **Stable multigerms in $\mathcal{O}(3,3)$**

Name	A_k	*Normal form*
Ordinary double points	$A_{1,1}$	$\{(x,y,z^2);$ $(x,y^2,z)\}$
Normal crossing of a plane with cuspidal edge	$A_{1,2}$	$\{(x,y^2,z);$ $(x,y,z^3+xz)\}$
Ordinary triple points	$A_{1,1,1}$	$\{(x,y,z^2);$ $(x,y^2,z);$ $(x^2,y,z)\}$

Remark 3.18. Let $f : (\mathbb{C}^3,0) \to (\mathbb{C}^3,0)$ be an $\mathcal{A}$-finitely determined map germ . According to Theorem 3.7 and the Tables 1.1 and 1.2 above, the germs of stable singularities appearing on the discriminant set of a stabilization of f are:

1. normal crossing of two planes, corresponding to the 1-dimensional curve of double points set, that are singularities of type $A_{1,1}$. Figure 5 b).

2. normal crossing of three planes, corresponding to the ordinary triple points. These singularities are of type $A_{1,1,1}$. Figure 6 c).

3. the 1-dimensional cuspidal edge, singularities of type A_2. Figure 5 a).

4. points of type A_3, also called swallowtail. Figure 6 a).

5. the transversal intersection of a cuspidal edge with a plane, that are points of type $A_{1,2}$. Figure 6 b).

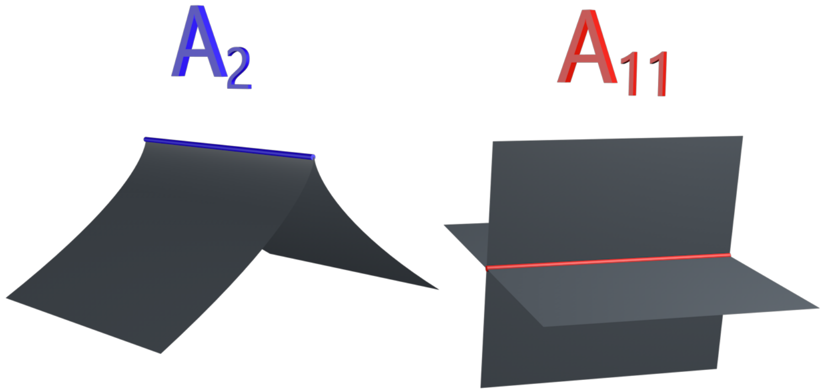


Figure 5: 1-stable singularities, $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$.

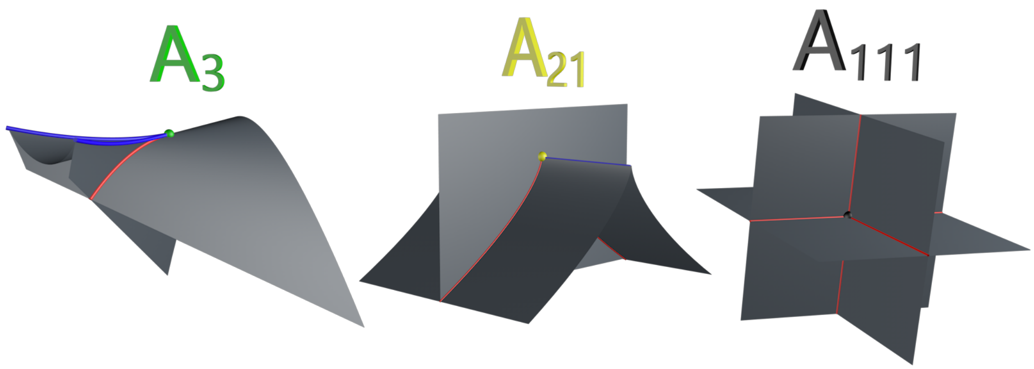


Figure 6: 0-stable singularities, $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$.

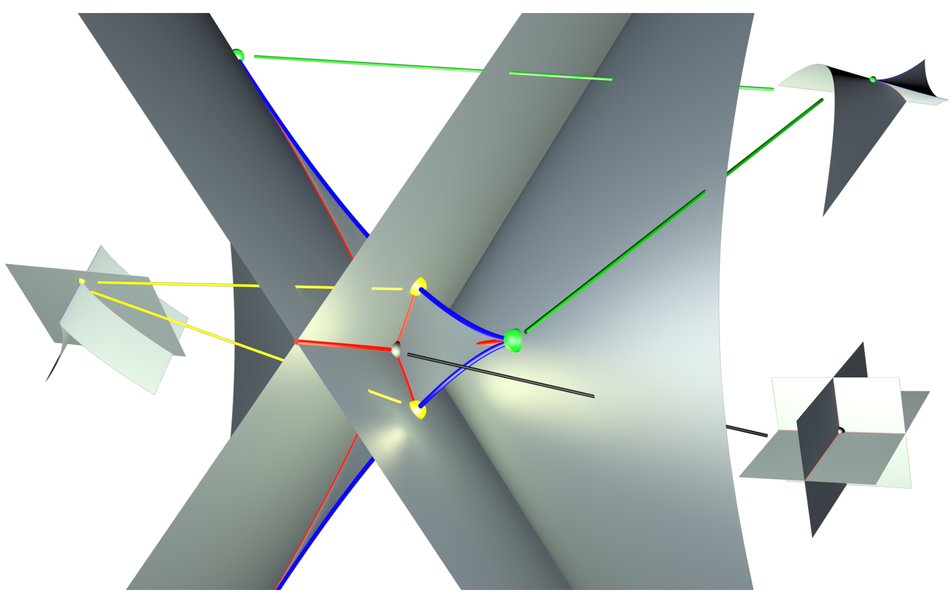


Figure 7: All real r-stable singularities, $(\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$.

In the Figure 7, we can see all real stable singularities appearing in discriminant set of one deformation of some $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$.

To obtain these results we describe the real geometry of this deformation, this means to stratify the parameters space according to the multiple points set that appear in the source and consequently the singularities that appear in the discriminant. For this, first we describe the ideals in the source that define all multiple point spaces. To obtain the strata in the target, we apply f in the strata of the source.

Consider the two parameters deformation of the germ $f(x, y, z) = (x, y, z^6 + yz + xz^2)$:

$$f_{u,v}(x, y, z) = (x, y, z^6 + yz + xz^2 + vz^4 + uz^3).$$

For (real) (u, v), we describe all singularities of $f_{u,v}$, and to investigate for which real parameters (u, v) we can find a germ $f_{u,v}$ that has critical points (in the source) whose image by $f_{u,v}$ form the 0-stable real singularities in the discriminant $\Delta(f_{u,v}) = f_{u,v}(\Sigma(f_{u,v}))$, and how many of them appear.

In a stabilization of this complex map germ

$$f(x, y, z) = (x, y, z^6 + yz + xz^2),$$

there are

$$\sharp A_3 = 3, \quad \sharp A_{1,2} = 6, \quad \sharp A_{1,1,1} = 1.$$

However, for all possible (u, v) we shall show that it is not possible to obtain all these ten 0-stable singularities in real coordinates.

Question: For a fixed pair (u, v), how many real isolated singularities appear in the discriminant of $f_{u,v}$?

Calculating the number of real A_3 (swallowtail)

```
> ring r=(0,u,v),(x,y,z),dp;
> idealsource([x,y,z6+yz+xz2+vz4+uz3],[3]);
//f:(C^3,0)--->(C^3,0) ; Partition = [3]

H[1]=36*z01^5+(4*v)*z01^3+y
H[2]=45*z01^4+(6*v)*z01^2-x
H[3]=20*z01^3+(4*v)*z01+(u)

//To access the ideal H, type:  setring MR; H;
```

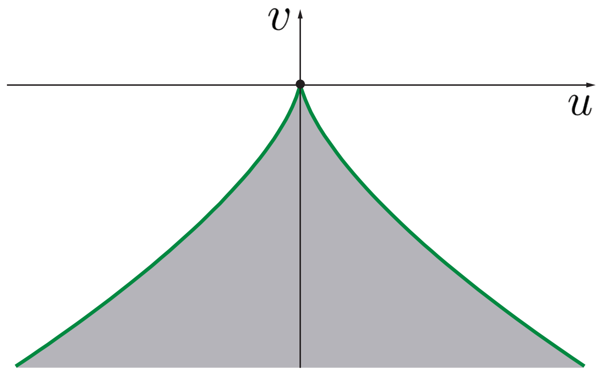
Renaming $z01 = z$, we have

$$\mathcal{I}^1(f, (3)) := \langle 36z^5 + 4vz^3 + y, 45z^4 + 6vz^2 - x, 20z^3 + 4vz + u\rangle.$$

Once found values for z, then x and y are obtained trivially by the first two generators. Then, it is enough to analyze the last generator: $\quad 20z^3 + 4vz + u = 0.$

$DA_3 := 64v^3 + 135u^2$ is the discriminant root of the polynomial $20z^3 + 4vz + u$.

In this case we always have real points A_3, but in different number, as we see in Figure 8.



	$\sharp A_3$ (real)
white	1
gray	3

Figure 8: $DA_3 := 64v^3 + 135u^2$.

Calculating the number of real $A_{1,1,1}$ (triple fold)

```
> ring r=(0,u,v),(x,y,z),dp;
> idealsource([x,y,z6+yz+xz2+vz4+uz3],[1,1,1]);
//f:(C^3,0)--->(C^3,0) ; Partition = [1,1,1]

H[1]=2*z01^2*z02^2*z03+2*z01^2*z02*z03^2+2*z01*z02^2*z03^2+y
H[2]=z01^2*z02^2+4*z01^2*z02*z03+4*z01*z02^2*z03+z01^2*z03^2+
      4*z01*z02*z03^2+z02^2*z03^2-x
H[3]=2*z01^2*z02+2*z01*z02^2+2*z01^2*z03+8*z01*z02*z03+2*z02^2*z03+
      2*z01*z03^2+2*z02*z03^2+(u)
H[4]=z01^2+4*z01*z02+z02^2+4*z01*z03+4*z02*z03+z03^2+(-v)
H[5]=2*z01+2*z02+2*z03

//To access the ideal H, type:  setring MR; H;
```

Renaming $z01 = z, z02 = z2, z03 = z3$, we have

$\mathcal{I}^3(f,(1,1,1)) := \langle 2z^2z_2^2z_3 + 2z^2z_2z_3^2 + 2zz_2^2z_3^2 + y, z^2z_2^2 + 4z^2z_2z_3 + z^2z_3^2 + 4zz_2^2z_3 + 4zz_2z_3^2 + z_2^2z_3^2 - x, 2z^2z_2 + 2z^2z_3 + 2zz_2^2 + 8zz_2z_3 + 2zz_3^2 + 2z_2^2z_3 + 2z_2z_3^2 + u, z^2 + 4zz_2 + 4zz_3 + z_2^2 + 4z_2z_3 + z_3^2 - v, 2z + 2z_2 + 2z_3\rangle$.

The resultant of the **fourth** and **fifth** generators of $\mathcal{I}^3(f,(1,1,1))$ in relation to z_3, is:

$$re1 := 8z^2 + 8zz_2 + 8z_2^2 + 4v.$$

The resultant of the **third** and **fifth** generators of $\mathcal{I}^3(f,(1,1,1))$ in relation to z_3, is:

$$re2 := -8z^2z_2 - 8zz_2^2 + 4u.$$

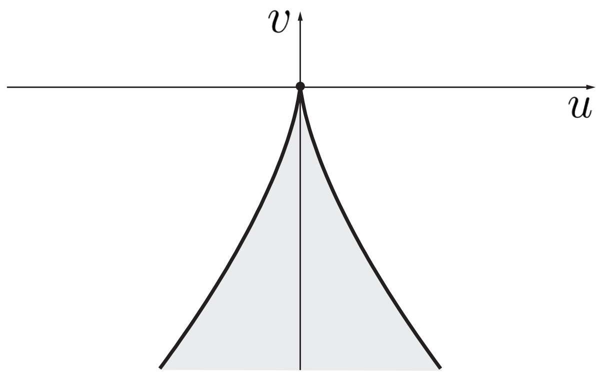
The resultant of $re1$ and $re2$ in relation to z_2, is:

$$re3 := 4096z^6+4096vz^4+4096uz^3+1024v^2z^2+2048uvz+1024u^2 = 1024(2z^3+vz+u)^2.$$

(Which depends only of variable z).

Thus, $\sharp A_{1,1,1}$ depends on the roots of $2z^3 + vz + u$ and $DA_{1,1,1} := 2v^3 + 27u^2$ is the discriminant root of polynomial $2z^3 + vz + u$.

In this case, we always have real $A_{1,1,1}$ points, see Figure 9.



	$\sharp A_{1,1,1}$ (real)
white	0
gray	1
black	0

Figure 9: $DA_{1,1,1} := 2v^3 + 27u^2$.

Calculating the number of real $A_{1,2}$ (fold-cusp)

```
> ring r=(0,u,v),(x,y,z),dp;
> idealsource([x,y,z6+yz+xz2+vz4+uz3],[1,2]);
//f:(C^3,0)--->(C^3,0) ; Partition = [1,2]

H[1]=6*z01^3*z02^2+12*z01^2*z02^3+6*z01*z02^4-y
H[2]=6*z01^3*z02+18*z01^2*z02^2+18*z01*z02^3+3*z02^4+x
H[3]=2*z01^3+12*z01^2*z02+18*z01*z02^2+8*z02^3+(-u)
H[4]=3*z01^2+6*z01*z02+6*z02^2+(v)

//To access the ideal H, type:  setring MR; H;
```

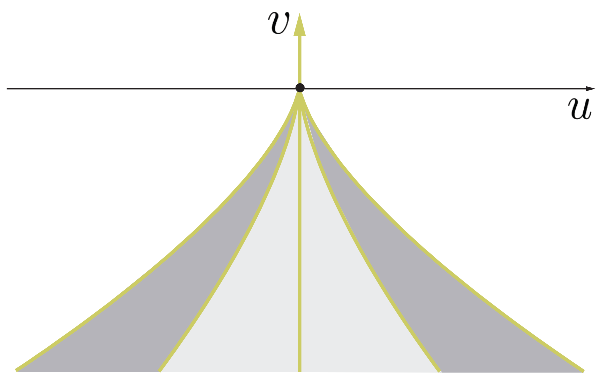
Renaming $z01 = z, z02 = z2$, we have

$$\begin{aligned}\mathcal{I}^2(f,(1,2)) \;\; := \;\; & \langle 6z^3z_2^2 + 12z^2z_2^3 + 6zz_2^4 - y, 6z^3z_2 + 18z^2z_2^2 + 18zz_2^3 + 3z_2^4 + x, \\ & 2z^3 + 12z^2z_2 + 18zz_2^2 + 8z_2^3 - u, 3z^2 + 6zz_2 + 6z_2^2 + v\rangle.\end{aligned}$$

The resultant of the **third** and **fourth** generators of $\mathcal{I}^2(f,(1,2))$ in relation to z_2, is:

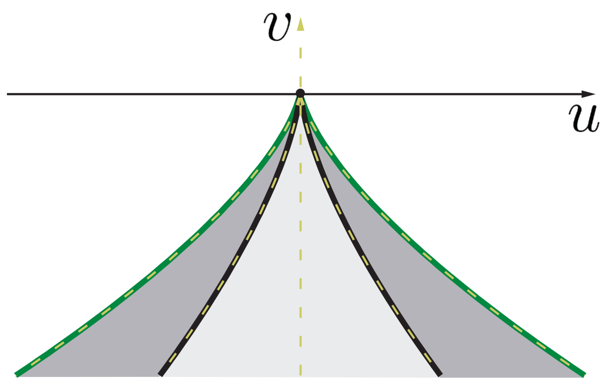
$$re := 1080z^6 + 1296vz^4 + 864uz^3 + 504v^2z^2 + 432uvz + 64v^3 + 216u^2.$$

$DA_{1,2} := u(64v^3 + 135u^2)(2v^3 + 27u^2)$ is the discriminant root of polynomial re. In this case we have maximum of four real points of type $A_{1,2}$, see Figure 10.



	$\sharp A_{1,2}$ (real)
white	0
dark gray	2
gray	4

Figure 10: $DA_{1,2} := u(64v^3 + 135u^2)(2v^3 + 27u^2)$.



	$\sharp A_3$	$\sharp A_{1,1,1}$	$\sharp A_{1,2}$
white	1	0	0
dark gray	3	0	2
gray	3	1	4
line $u = 0, v < 0$	3	1	3

Figure 11: Bifurcation set of all 0-stable singularities.

Thus, if parameters (u, v) are in region of color gray of Figure 11, we have the maximum number $8 < 10$ of real isolated singularities.

Real topology of the discriminant of $f(x, y, z) = (x, y, z^6 + yz + xz^2)$

In this part we describe the topological properties of the discriminant of deformation $f_{u,v}(x, y, z) = (x, y, z^6 + yz + xz^2 + uz^3 + vz^4)$. We show how to obtain the rank of the second homology of the discriminant using its Euler characteristic, moreover show explicitly the polyhedra which form this homology whose vertices are the 0-singularities of the discriminant of $f_{u,v}$.

First we list all definition ideals of r-stable types ($r = 0, 1, 2$) singularities of f in the source. If one want to obtain the definition ideals of the r-stable types in the target, use `eliminate` command in Singular or Maple or other program of your preference. In chapter 5, we will show another way to obtain the multiple point spaces in the target using presentation matrix and Fitting ideals. We remark that in this text, the curves in blue color are the cuspidal edges and curves in red color are double points curve. Points in black color are points of type $A_{1,1,1}$, points in yellow color are of type $A_{1,2}$ and points in green color are points of type A_3.

1. The 2-dimensional stable types in the source and in the target are respectively the critical and discriminant set:

   $\Sigma(f_{u,v}) = V\langle 6z^5 + 4vz^3 + 3uz^2 + 2xz + y\rangle.$

   $\Delta(f_{u,v}) := f(\Sigma(f_{u,v})) = V\langle 46656Z^5 + 62208XvZ^4 + (-13824v^3 + 34992u^2)Z^4 + 17280v^2X^2Z^3 - 46656uv^2YZ^3 + 13824X^3Z^3 + 77760uXYZ^3 + 32400vY^2Z^3 + (-9216v^4 - 3888u^2v)XZ^3 + (1024v^6 + 8640u^2v^3 + 8748u^4)Z^3 + 9216vX^4Z^2 - 3456uvX^2YZ^2 - 6480v^2XY^2Z^2 + 43200X^2Y^2Z^2 + 27000uY^3Z^2 + (-4352v^3 - 8640u^2)X^3Z^2 + (512v^5 + 8208u^2v^2)X^2Z^2 + (-5760uv^3 + 21384u^3)XYZ^2 + (-192v^4 - 27540u^2v)Y^2Z^2 + (-576u^2v^4 - 4860u^4v)XZ^2 + (768uv^5 + 5832u^3v^2)YZ^2 + (108u^4v^3 + 729u^6)Z^2 - 512v^2X^5Z + 2496uv^2X^3YZ + 1024X^6Z - 6912uX^4YZ + 10560vX^3Y^2Z - 19800uvXY^3Z - 1500v^2Y^4Z + 22500XY^4Z + (64v^4 + 576u^2v)X^4Z + (-4816v^3 + 9720u^2)X^2Y^2Z + (-16u^2v^3 - 108u^4)X^3Z + (-320uv^4 - 2808u^3v)X^2YZ + (576v^5 + 4536u^2v^2)XY^2Z + (120uv^3 + 1350u^3)Y^3Z + (72u^3v^3 + 486u^5)XYZ + (-24u^2v^4 - 162u^4v)Y^2Z - 128X^4Y^2v^2 + 560X^2Y^3uv^2 + 256X^5Y^2 - 1600X^3Y^3u + 2000X^2Y^4v - 3750Y^5uv + 3125Y^6 + (16v^4 + 144u^2v)X^3Y^2 + (-900v^3 + 2250u^2)XY^4 + (-4u^2v^3 - 27u^4)X^2Y^2 + (-72uv^4 - 630u^3v)XY^3 + (108v^5 + 825u^2v^2)Y^4 + (16u^3v^3 + 108u^5)Y^3\rangle.$

2. The 1-dimensional stable types in the source are:

   $D^1(f_{u,v}, (2)) = V\langle 24z^5 + 8vz^3 + 3uz^2 - y, 15z^4 + 6vz^2 + 3uz + x\rangle.$

   $D^2(f_{u,v}, (1,1)) = V\langle 6z^4z_2 + 12z^3z_2^2 + 12z^2z_2^3 + 6zz_2^4 + 2vz^2z_2 + 2vzz_2^2 + y, 3z^4 + 12z^3z_2 + 15z^2z_2^2 + 12zz_2^3 + 3z_2^4 + vz^2 + 4vzz_2 + vz_2^2 - x, 4z^3 + 6z^2z_2 + 6zz_2^2 + 4z_2^3 + 2vz + 2vz_2 + u\rangle.$

   eliminating the variable z_2 we obtain

   $D^2_1(f_{u,v}, (1,1)) = V\langle y + 2zx + 3z^2u + 4z^3v + 6z^5, 256x^3 - 27u^4 + 144u^2vx + 1152zux^2 - 128v^2x^2 + 108zu^3v - 4u^2v^3 + 1944z^2u^2x - 96zuv^2x + 16v^4x + 1344z^2vx^2 + 972z^3u^3 + 288z^2u^2v^2 + 16zuv^4 + 5040z^3uvx - 336z^2v^3x + 2448z^4x^2 + 4860z^4u^2v - 16z^3uv^3 + 32z^2v^5 + 8208z^5ux + 2256z^4v^2x + 6912z^6u^2 + 5904z^5uv^2 - 112z^4v^4 + 9936z^6vx + 19008z^7uv + 1712z^6v^3 + 8640z^8x + 15120z^9u + 11088z^8v^2 + 19440z^{10}v + 10800z^{12}\rangle.$

3. The 0-dimensional stable types in the source are:

   $D^1(f_{u,v}, (3)) = V\langle 36z^5 + 4vz^3 + y, 45z^4 + 6vz^2 - x, 20z^3 + 4vz + u\rangle.$

   $D^2(f_{u,v}, (1,2)) = V\langle 6z^3z_2^2 + 12z^2z_2^3 + 6zz_2^4 - y, 6z^3z_2 + 18z^2z_2^2 + 18zz_2^3 + 3z_2^4 + x, 2z^3 + 12z^2z_2 + 18zz_2^2 + 8z_2^3 - u, 3z^2 + 6zz_2 + 6z_2^2 + v\rangle.$

$D^3(f_{u,v},(1,1,1)) = V\langle 2z^2z_2^2z_3 + 2z^2z_2z_3^2 + 2zz_2^2z_3^2 + y, z^2z_2^2 + 4z^2z_2z_3 + z^2z_3^2 + 4zz_2^2z_3 + 4zz_2z_3^2 + z_2^2z_3^2 - x, 2z^2z_2 + 2z^2z_3 + 2zz_2^2 + 8zz_2z_3 + 2zz_3^2 + 2z_2^2z_3 + 2z_2z_3^2 + u, z^2 + 4zz_2 + 4zz_3 + z_2^2 + 4z_2z_3 + z_3^2 - v, 2z + 2z_2 + 2z_3\rangle.$

eliminating the variables z_2 and z_3 we obtain

$D_1^2(f_{u,v},(1,2)) = V\langle 12x + 18zu + v^2 + 30z^2v + 45z^4, 6y - zv^2 - 6z^3v - 9z^5, 27u^2 + 54zuv + 8v^3 + 108z^3u + 63z^2v^2 + 162z^4v + 135z^6\rangle.$

$D_1^3(f_{u,v},(1,1,1)) = V\langle u + zv + 2z^3, 4x - v^2, 2y + zv^2 + 2z^3v\rangle \subset \mathbb{C}^3.$

Case: 8 real points in the target

Take parameters $0 \neq u, v$ in gray color region in Figure 11. Let V = vertices, A = lines, F = faces. For this case we have $8V$; $15A$; $10F$.

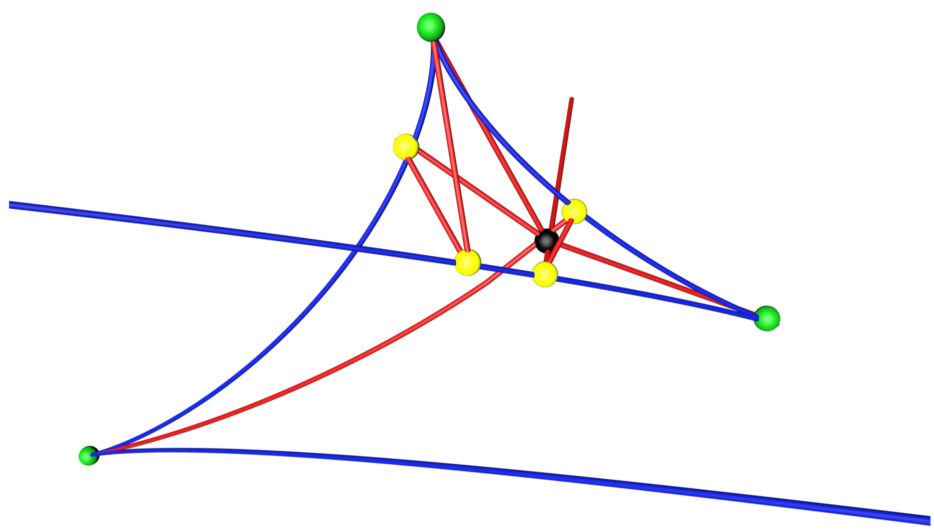


Figure 12: Case with 8 real points in the target.

Therefore the Euler characteritic of $\Delta(f_{u,v})$ is:

$$\chi(\Delta(f_{u,v})) = V - A + F = 8 - 15 + 10 = 3.$$

On the other side,

$$\chi(\Delta(f_{u,v})) = \text{rank } H_0(\Delta(f_{u,v})) - \text{rank } H_1(\Delta(f_{u,v})) + \text{rank } H_2(\Delta(f_{u,v})).$$

As $\Delta(f_{u,v})$ is **simply connected**, then rank $H_0(\Delta(f_{u,v})) = 1$ and rank $H_1(\Delta(f_{u,v})) = 0$. Therefore,

$$3 = 1 - 0 + \text{rank } H_2(\Delta(f_{u,v})),$$

and we easily obtain rank $H_2(\Delta(f_{u,v})) = 2$, or in other words, 2 spheres in the bouquet. We remark that is possible to obtain only 2 polyhedra, or 2 spheres, Figure 12.

Case: 7 real points in the target

Take parameters u, v in line $u = 0, v < 0$ in Figure 11. In this case we have $7V$, $14A$, $11F$.

Therefore the Euler characteritic of $\Delta(f_{u,v})$ is:

$$\chi(\Delta(f_{u,v})) = V - A + F = 7 - 14 + 11 = 4.$$

On the other side,

$$\chi(\Delta(f_{u,v})) = \text{rank } H_0(\Delta(f_{u,v})) - \text{rank } H_1(\Delta(f_{u,v})) + \text{rank } H_2(\Delta(f_{u,v})).$$

As $\Delta(f_{u,v})$ is **simply connected**, then rank $H_0(\Delta(f_{u,v})) = 1$ and rank $H_1(\Delta(f_{u,v})) = 0$. Therefore,

$$4 = 1 - 0 + \text{rank } H_2(\Delta(f_{u,v})),$$

and

$$\text{rank } H_2(\Delta(f_{u,v})) = 3,$$

or in other words, we have 3 spheres in the bouquet.

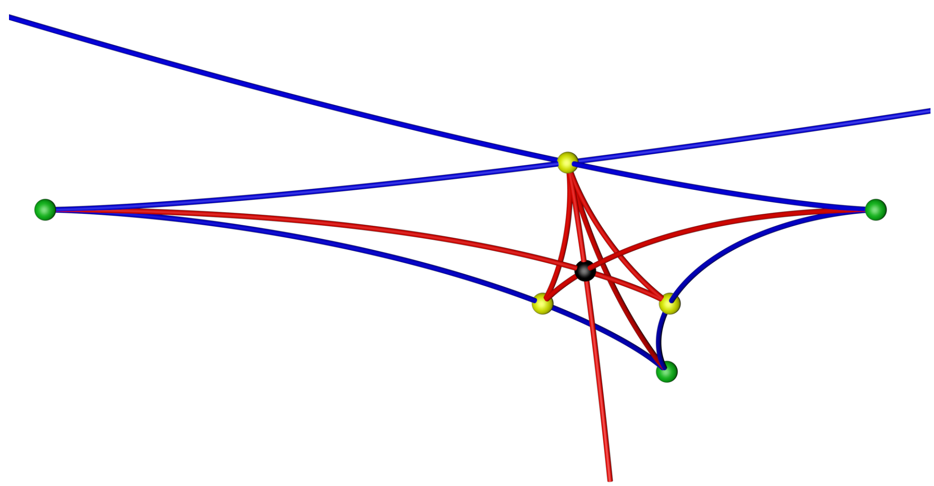


Figure 13: $7V$; $14A$; $11F$.

With 8 real points in the target, we have a wedge of 2 **spheres**.

With 7 real points in the target, we have a wedge of 3 **spheres**.

With $p < 7$ real points in the target, the number of spheres in the bouquet is less than two.

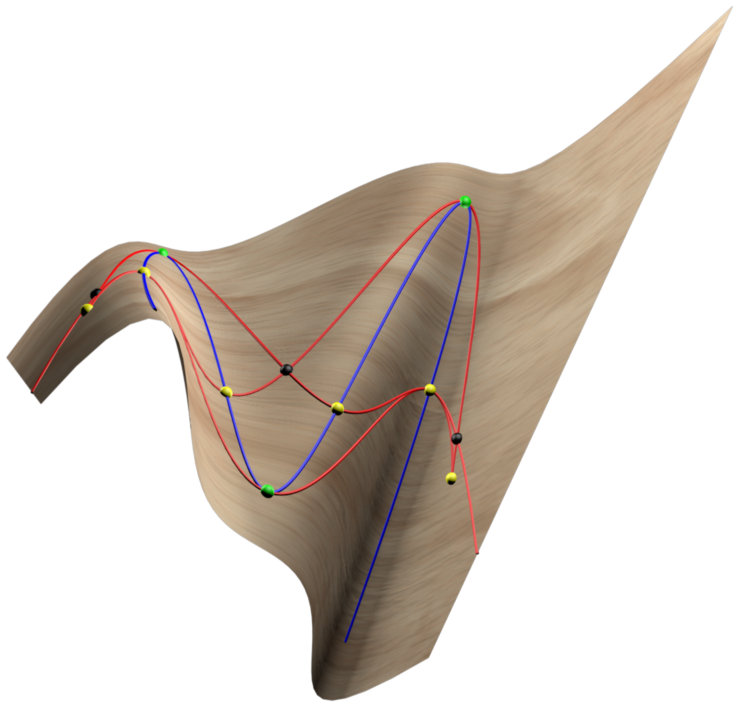


Figure 14: $\Sigma(f_{u,v})$, $u = 0, v < 0$.

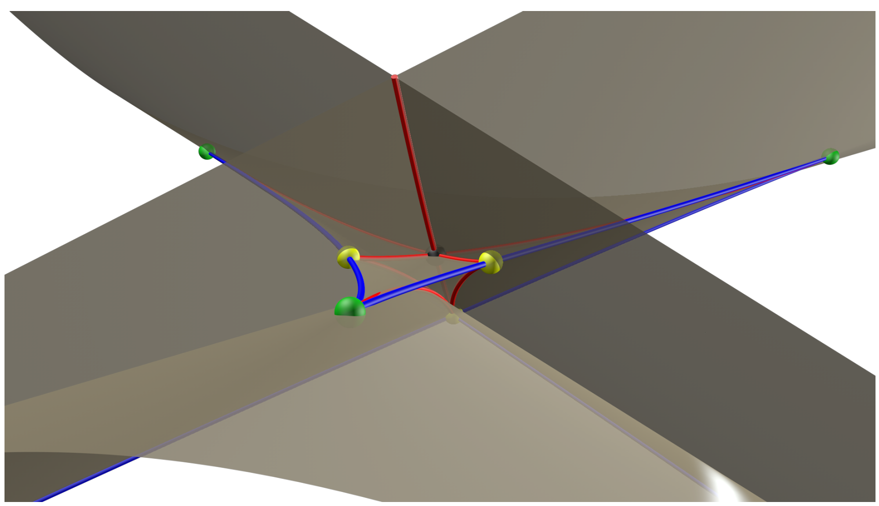


Figure 15: $\Delta(f_{u,v})$, $u = 0, v < 0$.

4 Multiple point spaces in the target via Fitting ideals

The multiple point spaces of a map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$ with $n \geq p$ play an important role in the study of its geometry, as well as the topology of the image of a stable perturbation. The kth target multiple points space $M_k(f)$ is the closure in the image of the set of points having k or more preimages, counting multiplicities.

When $f : \mathcal{X} \to \mathcal{Y}$ is a finite analytic map of complex manifolds, the space $M_k(f)$ has a natural analytic structure as the subspace of $\mathcal{Y}$ defined by the $(k-1)$'st Fitting ideal $\mathcal{F}_{k-1}(f_*\mathcal{O}_{\mathcal{X}})$ of the pushforward $f_*\mathcal{O}_{\mathcal{X}}$. This structure is good when $\dim \mathcal{Y} = \dim \mathcal{X} + 1$, $\mathcal{X}$ Cohen-Macaulay and $\mathcal{Y}$ smooth.

In [Mond-Pellikaan, [33]] is described an algorithm to compute a presentation of the pushforward module $f_*\mathcal{O}_{\mathcal{X}}$ for a finite map germ $f\colon \mathcal{X} \to (\mathbb{C}^{n+1}, 0)$, where $\mathcal{X}$ is Cohen-Macaulay of dimension n. The application of this algorithm in general is not an easy task and it is a challenge to find the presentation matrix, even in simple cases, because it is necessary the computation of Groebner bases. For the case $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ of corank 1, as $\mathcal{X} = \Sigma(f)$ is an n-dimensional Cohen-Macaulay variety, then we can apply the algorithm of Mond-Pellikan to obtain a presentation. For this case with f in its pre-normal form it is possible to obtain a presentation without use Groebner bases, [Miranda-Saia, [30]], and moreover we present a fast implementation in Maple and Singular of the algorithm of Mond and Pellikaan, showing explicitly how to compute the elements of the polynomial presentation matrices for such maps.

For general maps $f\colon \mathcal{X} \to (\mathbb{C}^{n+1}, 0)$, where $\mathcal{X}$ is Cohen-Macaulay of dimension n, [Hernandes, Miranda, Peñafort-Sanchis, [13]], describe an algorithm and implementation in Singular to compute a presentation of the pushforward module $f_*\mathcal{O}_{\mathcal{X}}$ for a finite map germ. The algorithm is based on a method by [Mond and Pellikaan,[33]], but introduces an improvement which allows to circumvent certain problems, concerning the limitation to polynomial inputs and outputs of commutative algebra systems, such as Singular [5]. As we will see, this improvement also makes the algorithm more efficient from a computational point of view. The reader can find in [Miranda and Peñafort-Sanchis,[31]] a Singular library containing an implementation of the algorithm.

In this chapter we give some applications to problems in singularity theory, computed by means of an implementation, called presmatrix, of presentation matrices in the software Singular. We show the computation of target and source multiple-point schemes for map germs $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, discriminants and certain topological invariants of maps – leading, for example, to the answer of a question, due to Gaffney and Mond, about the topological classification for corank 2 map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$. All these applications are based on *Fitting ideals*, which play a crucial role in the theory of singularities of map germs and in enumerative geometry (see for instance [Kleiman, Lipman and Ulrich,[20]].

It is worth saying that the Singular implementation of the algorithm has already been used in the works of other authors: In [Ballesteros, Oréfice-Okamoto and Tomazella, [35]], the authors use it to compute discriminant curves of map germs from a complete intersection surface to the plane. Oset Sinha, Ruas and Wik Atique have used the algorithm to compute the image of a stable map germ of corank 2 from $\mathbb{C}^8$ to $\mathbb{C}^9$, which plays an important role in their work on the extra-nice dimensions [Oset Sinha, Ruas and Wik Atique, [37]]. Recently, O. N. Silva, [42], used the algorithm to compute source double points of certain maps as in Section 4.3.3, obtaining the first known counter-example to a conjecture by M. A. S. Ruas, on the equivalence between Whitney equisingularity and Topological triviality.

In this chapter we will describe and apply these algorithms with several examples.

For convenience to the reader, we describe Mond-Pellikaan's original method to obtain presentation matrices.

Background for presentation matrix

Let M be a module over a commutative unitary ring R. A *presentation* of M is an exact sequence

$$R^p \xrightarrow{\lambda} R^q \xrightarrow{\psi} M \longrightarrow 0.$$

If M admits a presentation, then we say that M is a *finitely presented* module, and any matrix Λ associated to λ is called a *presentation matrix* of M. It is well known that any finitely generated module over a Noetherian ring is finitely presented (see [10]).

If M is a finitely presented module as above, then its k-th Fitting ideal is given by

$$F_k(M) = \begin{cases} 0 & \text{if } \ k < 0; \\ \langle \text{minors of order } q-k \text{ of } \Lambda \rangle & \text{if } \ 0 \leq k < \min(q,p); \\ R & \text{if } \ \min(q,p) \leq k. \end{cases}$$

Fitting ideals are invariant under module isomorphisms and they do not depend on the chosen presentation of M (see [21]).

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a finite map germ and $\mathcal{O}_\mathcal{X}, \mathcal{O}_\mathcal{Y}$ the rings of holomorphic functions of $\mathcal{X}$ and $\mathcal{Y}$. The *pushforward module* $f_*\mathcal{O}_\mathcal{X}$ is just $\mathcal{O}_\mathcal{X}$, regarded as an $\mathcal{O}_\mathcal{Y}$-module via f. Finiteness of f implies that $f_*\mathcal{O}_\mathcal{X}$ is finite, and hence finitely presented. For simplicity, we write the corresponding Fitting ideals as $\mathcal{F}_k(f) = \mathcal{F}_k(f_*\mathcal{O}_\mathcal{X})$. As shown by [Mond and Pellikaan, [33]], the kth Fitting ideal of $f_*\mathcal{O}_\mathcal{X}$ defines the $(k+1)$th multiple points space of f in $\mathcal{Y}$, which we write as $M_k(f) = V(\mathcal{F}_{k-1}(f))$.

4.1 Mond-Pellikaan algorithm for constructing a presentation

Let $\mathcal{X}$ be an n-dimensional germ of Cohen Macaulay space, and let

$$f\colon \mathcal{X} \to (\mathbb{C}^{n+1}, 0)$$

satisfy the following extra condition: If we let

$$\tilde{f}\colon \mathcal{X} \to (\mathbb{C}^n, 0),$$

be the germ obtained by composing f with the projection from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^n, 0)$ which forgets the last coordinate, then $\tilde{f}$ is a finite map germ. Since our interest is to obtain a computer implementation, we are going to switch from the holomorphic setting of Mond and Pellikaan to the rational setting software like SINGULAR can handle. This is mostly a matter of language, and the results of Mond and Pellikaan apply here exactly in the same way.

Let $A = \mathbb{C}[X,Y]_{\langle X,Y\rangle}$ be the localization at the maximal ideal at the origin of the ring of polynomials in the $n+1$ variables $X = X_1,\ldots,X_n$ and Y. We denote $\tilde{A} = \mathbb{C}[X]_{\langle X\rangle}$ and $B = (\mathbb{C}[x]/I)_{\langle x\rangle}$, with variables $x = x_1,\ldots,x_\ell$, and assume that B is a Cohen-Macaulay ring of dimension n. Let

$$\phi\colon A \to B$$

be a morphism of local rings given by $X_i \mapsto f_i, i = 1,\ldots,n$ and $Y \mapsto f_{n+1}$, for some polynomials $f_j \in \mathbb{C}[x]$. Write

$$\tilde{\phi}\colon \tilde{A} \to B$$

for the restricted morphism, and assume that B is minimally generated by $g_1,\ldots,g_h$ as an $\tilde{A}$-module. Since B is generated by $g_1,\ldots,g_h$, there exist $\alpha_{ij} \in \tilde{A}$, $1 \leq i,j \leq h$, satisfying the equations

$$Yg_i = \sum_{j=1}^{h} \alpha_{ij} g_j, \text{ for every } 1 \leq i \leq h. \tag{4.2}$$

Let $\lambda : A^h \to A^h$ be given by multiplication by the matrix Λ whose entries are

$$\Lambda_{ij} = \alpha_{ij} - \delta_{ij} Y,$$

where δ_{ij} stands for the Kronecker delta function.

If $\psi\colon A^h \to B$ is the epimorphism given by $e_i \mapsto g_i$, where $e_i \in A^h$ is the element whose only non-zero entry is 1 in the ith position, then the inclusion Im $\lambda \subseteq$ Ker ψ follows from (4.2). Mond and Pellikaan show that indeed the sequence

$$A^h \xrightarrow{\lambda} A^h \xrightarrow{\psi} B \longrightarrow 0 \tag{4.3}$$

is exact [33]. Therefore, the matrix

$$\Lambda = \begin{pmatrix} \alpha_{11} - Y & \alpha_{12} & \cdots & \alpha_{1h} \\ \alpha_{21} & \alpha_{22} - Y & \cdots & \alpha_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{h1} & \alpha_{h2} & \cdots & \alpha_{hh} - Y \end{pmatrix}$$

is a presentation matrix for B.

Example 4.1. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be the cross-cap map (Figure 16), given by

$$(x,y) \mapsto (x, y^2, xy).$$

Let $A = \mathbb{C}[X_1,X_2,Y]_{\langle X_1,X_2,Y\rangle}$, $\tilde{A} = \mathbb{C}[X_1,X_2]_{\langle X_1,X_2\rangle}$ and $B = \mathbb{C}[x,y]_{\langle x,y\rangle}$ and consider the ring homomorphism $A \to B$ given by $X_1 \mapsto x, X_2 \mapsto y^2$ and $Y \mapsto xy$.

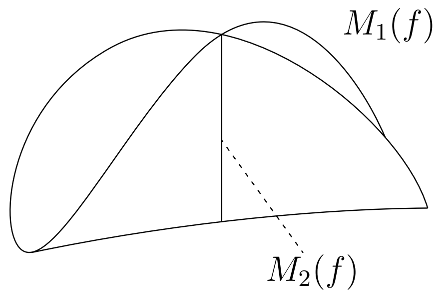


Figure 16: Image of a cross-cap (Whitney umbrella).

Observe that B is an $\tilde{A}$-module minimally generated by $g_1 = 1$ and $g_2 = y$, and thus we can apply Mond-Pellikaan algorithm. From the equalities

$$Y \cdot g_1 = xy \cdot g_1 = x \cdot g_2 = X_1 \cdot g_2 \quad \text{and} \quad Y \cdot g_2 = xy \cdot g_2 = xy^2 \cdot g_1 = X_1 X_2 \cdot g_1$$

we obtain the presentation matrix

$$\Lambda = \begin{pmatrix} -Y & X_1 \\ X_1 X_2 & -Y \end{pmatrix}.$$

The matrix Λ can be used to compute the Fitting ideals of $f_* \mathcal{O}_2$, which determined the multiple-point schemes in the target of f. The image of f and the double points space are, respectively:

$$M_1(f) = V(Y^2 - X_1^2 X_2) \quad \text{and} \quad M_2(f) = V(X_1, Y).$$

4.2 Algorithm for polynomial presentation matrices

Computer algebra system such as Singular only admit polynomial inputs and outputs. In this section we deal with the problem of how to find presentation matrices whose entries are polynomial. We start with the following trivial remark:

Remark 4.2. *The elements $\alpha_{ij} \in \tilde{A}$ in (4.2) are fractions $\alpha_{ij} = a_{ij}/b_{ij}$, for some polynomials $a_{ij}, b_{ij} \in \mathbb{C}[X]$ and $b_{ij}(0) \neq 0$. Multiplying the ith row by the least common multiple of the elements $b_{ij},\ j = 1, \ldots, h$, we obtain another presentation matrix, whose entries are polynomial.*

The previous remark guarantees, given a minimal collection of generators g_i, the existence of a polynomial presentation matrix is of the following form:

Definition 4.3. *Given $g_1, \ldots, g_h \in B$, an* **MP**-matrix *(for g_i) is a matrix*

$$\Lambda = \begin{pmatrix} \beta_{11} - u_1 Y & \beta_{12} & \cdots & \beta_{1h} \\ \beta_{21} & \beta_{22} - u_2 Y & \cdots & \beta_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{h1} & \beta_{h2} & \cdots & \beta_{hh} - u_h Y \end{pmatrix},$$

with $\beta_{ij}, u_j \in \mathbb{C}[X], u_j(0) = 1$*, such that* $u_i g_i Y = \sum_{j=1}^{h} \beta_{i,j} g_j$ *for* $1 \leq i \leq h$*.*

With the previous notations, let $g_1, \ldots, g_h$ be a minimal system of generators of B as an $\tilde{A}$-module. It follows from Remark 4.2 that B admits an **MP**-matrix Λ for $g_1, \ldots, g_h$ as a presentation matrix. The matrix is given by some polynomials $\beta_{ij}, u_j \in \mathbb{C}[X], u_j(0) = 1$, satisfying the conditions

$$\phi(u_j Y) g_j \equiv \sum_{i=1}^{h} \phi(\beta_{ij}) g_i \mod I, \text{ for all } 1 \leq j \leq h.$$

To find the jth row of such Λ, one fixes polynomials up to some degree d:

$$\beta_{ij} = \sum_{|\alpha| \leq d} a_{i,\alpha} X^{\alpha} \text{ for } i = 1, \ldots, h,$$

$$u_{j,\alpha} = 1 + \sum_{1 \leq |\alpha| \leq d} b_{\alpha} X^{\alpha},$$

and tries to find $a_{i,\alpha}, b_{\alpha} \in \mathbb{C}$, such that the polynomial

$$P_d(x) = \phi(u_j Y) g_j - \sum_{i=1}^{h} \phi(\beta_{ij}) g_i$$

reduces to 0 modulo I. This is a linear system on $a_{i,\alpha}, b_{\alpha}$, prescribed by the vanishing of the coefficient of each x^{α} in the reduction of P_d modulo I. If that is not possible for the degree d, then one increases d and starts all over again.

Due to the reduction process, and to the clearing of denominators in Remark 4.2, there is no obvious way to estimate the degrees of the entries in an **MP**-matrix Λ in terms of the degrees of f_i and the generators of I. Unfortunately, the usage of reductions, and the increasing number of parameters $a_{i,\alpha}, b_{\alpha}$ involved, make the complexity of the procedure explained above grow very rapidly as d increases. In order to keep the degree d as low as possible, it seems a good idea to consider a class of matrices bigger than the set of **MP**-matrices. The following example illustrates this situation.

Example 4.4. Let $\mathcal{X} = V(I) \subseteq \mathbb{C}^3$, with $I = \langle z - x^k y \rangle$, and let $f\colon \mathcal{X} \to \mathbb{C}^3$ be given by

$$(x, y, z) \mapsto (x, y^2 + xz, z).$$

In our usual setting $A = \mathbb{C}[X_1, X_2, Y]_{\langle X_1, X_2, Y \rangle}, \widetilde{A} = \mathbb{C}[X_1, X_2]_{\langle X_1, X_2 \rangle}, B = \left(\frac{\mathbb{C}[x,y,z]}{\langle z - x^k y \rangle} \right)_{\langle x,y,z \rangle}$, ϕ is given by $X_1 \mapsto x, X_2 \mapsto y^2 + xz$ and $Y \mapsto z$, and the pushforward module B is minimally generated by $g_1 = 1$ and $g_2 = y$ as an $\tilde{A}$-module. It is easy to check that the matrix

$$\Lambda = \begin{pmatrix} Y & -X_1^k \\ -X_1^k X_2 & Y + X_1^{2k+1} \end{pmatrix}$$

is an **MP**-matrix for g_1, g_2. However, the matrix

$$\Lambda' = \begin{pmatrix} Y & -X_1^k \\ X_1^k (X_1 Y - X_2) & Y \end{pmatrix}$$

is also a presentation matrix (by Theorem 4.7 below). The procedure explained before needs to consider polynomials up-to degree $2k+1$ in order to find Λ, but a more flexible version, allowing matrices such as Λ', will stop at degree $k+2$.

Definition 4.5 ([13]). *With the previous notations, given $g_1,\dots,g_h \in B$, an* **HMP**-matrix (for g_i) *is a matrix Λ with polynomial entries in $\mathbb{C}[X,Y]$, satisfying the following conditions:*

C1. $\displaystyle\sum_{j=1}^{h} \phi(\Lambda_{ij})g_j \equiv 0 \mod I$, *for $i=1,\dots,h$.*

C2. $\Lambda_{ij}(0,Y) - \Lambda_{ij}(0,0) = \begin{cases} Y\cdot u_i(Y) \text{ if } i=j; \\ 0 \text{ if } i\neq j, \end{cases}$

where $u(Y)\in\mathbb{C}[Y]$ satisfying $u(0)\neq 0$.

Remark 4.6. *Every* **MP**-*matrix is an* **HMP**-*matrix. On the other hand, the* **HMP**-*matrix Λ' in Example 4.4 is not an* **MP**-*matrix.*

Theorem 4.7. *If Λ is an* **HMP**-*matrix for a minimal set of generators $g_1,\dots,g_h$ of B as an $\tilde{A}$-module, then Λ is a presentation matrix for B as an A-module.*

Proof The proof is similar to the one for Mond-Pellikaan's algorithm. Take the sequence of A-modules

$$A^h \xrightarrow{\Lambda} A^h \xrightarrow{\psi} B \longrightarrow 0,$$

where ψ is determined by $e_i \mapsto g_i$, and e_i the ith canonical vector in A^h. Condition **C1** implies $\operatorname{Im}\Lambda \subseteq \operatorname{Ker}\psi$, and ψ is an epimorphism, so it suffices to show that $\operatorname{Coker}\Lambda = B$.

Let $B' = (\mathbb{C}[x,t]/I)_{\langle x,t\rangle}$ and let $\phi'\colon A\to B'$ be given by $X_i \mapsto f_i, i=1,\dots,n$ and $Y\mapsto f_{n+1}+t$. From the fact that B is minimally generated by $g_1,\dots,g_h$ as an $\tilde{A}$-module, it follows that B' is a free A-module minimally generated by $g_1,\dots,g_h$. Let $\eta\colon A^h\to B'$ be the isomorphism given by

$$(a_1,\dots,a_h)\mapsto \sum_{i=1}^{h}\phi'(a_i)g_i$$

and let $\varphi\colon B'\to B'$ be the morphism defined by extending

$$g_j \mapsto \sum_{i=1}^{h}\phi'(\Lambda_{ij})\cdot g_i, \text{ for } j=1,\dots,h,$$

A-linearly, so that the diagram

$$\begin{array}{ccc} A^h & \xrightarrow{\Lambda} & A^h \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\ B' & \xrightarrow{\varphi} & B' \end{array}$$

commutes. We will show that $\operatorname{Coker}\varphi = B$.

The morphism φ extends A-linearly the assignations $g_j \mapsto \sum_{i=1}^{h} \phi'(\Lambda_{ij})g_i$, for $j = 1,\ldots,h$. Consider the expansion

$$\begin{aligned}
\sum_{i=1}^{h} \phi'(\Lambda_{ij}) \cdot g_i &= \sum_{i=1}^{h} (\Lambda_{ij}(f_1,\ldots,f_{n+1}+t)) \cdot g_i \\
&= \sum_{i=1}^{h} (\Lambda_{ij}(f_1,\ldots,f_{n+1})) \cdot g_i + \sum_{i=1}^{h} (\sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \Lambda_{ij}}{\partial Y^k}(f_1,\ldots,f_{n+1})t^k) \cdot g_i \\
&= t \sum_{i=1}^{h} (\sum_{k=1}^{\infty} \frac{1}{k!} \phi(\frac{\partial^k \Lambda_{ij}}{\partial Y^k}) t^{k-1}) \cdot g_i.
\end{aligned}$$

It follows that φ splits as the composition $B' \xrightarrow{\cdot t} B' \xrightarrow{\psi} B'$, where the first morphism is multiplication by t and ψ is obtained by extending $g_j \mapsto g'_j = \sum_i R_{ij} g_i$, with

$$R_{ij} = \sum_{k=1}^{\infty} \frac{1}{k!} \phi(\frac{\partial^k \Lambda_{ij}}{\partial Y^k}) t^{k-1}.$$

It suffices to show that ψ is an A-module isomorphism, that is, that $g'_1,\ldots,g'_h$ is a system of generators of B'. This is equivalent to show that the collection of the classes of $g'_1,\ldots,g'_h$ is a $\mathbb{C}$-basis $B'/\mathfrak{m}B'$, where $\mathfrak{m}$ is the maximal ideal in A. If $a \in A$ is divisible by some X_i, then $\phi(a) \in \mathfrak{m}B \subset \mathfrak{m}B'$. Therefore, condition **C2** implies that the classes of the non-diagonal coefficients $R_{ij}, i \neq j$, are all zero, and the classes of R_{ii} are all non-zero. This implies that the collection of classes of $g'_1,\ldots,g'_h$ is a basis, as desired.

Remark 4.8. *It is clear that Theorem 4.7 works for holomorphic maps as well. With our usual assumptions, if a holomorphic map $f\colon \mathcal{X} \to \mathbb{C}^{n+1}$ has polynomial coordinate functions and $g_1,\ldots,g_h$ are polynomial minimal set of generators of $\tilde{f}_*\mathcal{O}_X$, then any* **HMP**-*matrix for $g_1,\ldots,g_h$ is a presentation matrix for $f_*\mathcal{O}_X$.*

4.3 Implementation and applications

In this section we describe an algorithm to obtain a matrix Λ satisfying **C1** and **C2**, and we give some applications. An implementation of this algorithm in Singular can be found in [31].

We recall previous assumptions and notations: We use variables $X = X_1,\ldots,X_n$ and Y, and variables $x = x_1,\ldots,x_\ell$. We write $A = \mathbb{C}[X,Y]$ and $\tilde{A} = \mathbb{C}[X]_{\langle x \rangle}$. I is an ideal in $\mathbb{C}[x]_{\langle x \rangle}$, such that $B = \mathbb{C}[x]_{\langle x \rangle}/I$ is a Cohen Macaulay n-dimensional ring. We have a morphism of rings ϕ: $A \to B$, such that B is finitely generated as $\tilde{A}$-module. It is well known that a minimal set of generators $\{g_1,\ldots,g_h\}$ for B as $\tilde{A}$-module can be obtained as representatives of the elements of a basis of the $\mathbb{C}$-vector space $B/\mathfrak{m}B$, where $\mathfrak{m}$ is the ideal maximal ideal $\langle X_1,\ldots,X_n\rangle$ in $\tilde{A}$. We assume that such a basis can be computed by an internal procedure of the software used for implementation. We also assume the software to be able to perform Groebner bases computations, in particular the reduction of an ideal with respect to another one (see [10]). In Singular, this operations can be computed by using the instructions `kbase` and `reduce`, respectively.

The outline of our algorithm is as follows:

INPUTS: f and I.
Compute a $\mathbb{C}$-basis $\{g_1, g_2, \ldots, g_h\}$ of $B/\mathfrak{m}B$.
FOR $i = 1, \ldots, h$ do:
  DEFINE $w := 1$ and $k := 0$;
  WHILE $w \neq 0$ DO:
    $k := k + 1$;
    Consider $v_{i1}, \ldots, v_{ih}$, where
      $v_{ij} = \sum_{|\alpha| \leq k} a_{ij}^{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y^{\alpha_{n+1}}$ is a *generic polynomial* satisfying **(P)** (see Remark 4.9);
    Compute the reduction $P(a_{ij}^{\alpha}, x)$ of $\sum_{j=1}^{h} \phi(v_{ij}) g_j$ modulo I;
    IF there exists $\tilde{a}_{ij}^{\alpha} \in \mathbb{C}$, such that $P(\tilde{a}_{ij}^{\alpha}, x) = 0$
      THEN $\lambda_{ij} := \sum_{|\alpha| \leq k} \tilde{a}_{ij}^{\alpha} X^{\alpha}$ and $w := 0$;
OUTPUT: Matrix presentation $\Lambda = (\lambda_{ij})$.

PRESMATRIX: ALGORITHM TO COMPUTE AN HMP-MATRIX.

Remark 4.9. By a generic polynomial we mean that the coefficients a_{ij}^{α} are parameters in the base ring. Property **(P)** is as follows:

- $v_{ii}(0, \ldots, 0, Y) \equiv Y \mod (Y^2)$;
- $v_{ij}(0, \ldots, 0, Y) \in \mathbb{C}$, for all $j \neq i$.

By construction, condition **(P)** implies that Λ satisfies condition **C2**, and the fact that the reduction $P(\tilde{a}_{ij}^{\alpha}, X)$ vanishes ensures that **C1** holds. Note that the degree of the generic polynomials v_{ij} grows with the "while" loop, and the algorithm runs over all the matrices considered in Remark 4.2. Since we assumed that B is a finitely generated $\tilde{A}$-module, the algorithm terminates.

See Subsection 5.2 for a SINGULAR library implementation of the PRESMATRIX algorithm.

In subsections below we show some applications in singularity theory using PRESMATRIX. All computations showed here were done using a computer equipped with processor Intel Core i7-4790k, 4 Ghz, 32Gb Ram memory. We use standard singularity theory notation for which the reader can find the details in the references.

4.3.1 Topological invariants for maps from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$

Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ be a corank 2 map germ given by

$$(x, y) \longmapsto (xy, x^4 + y^{37} + x^2 y^{23}).$$

In this example we will calculate the following invariants of f: $\mu(\Sigma(f))$ Milnor number of critical set points, $\sharp A_2(f)$ number of cusps, $\sharp A_{1,1}(f)$ number of ordinary double points and $\mu(\Delta(f))$ Milnor number of discriminant curve.

```
> LIB  "sing.lib";   //library for calc. Milnor number
> ring r=0,(x,y),ds;
> ideal f=xy,x4+y37+x2y23;
```

```
> ideal Jf=det(jacob(f));   //jacobian ideal of f
> milnor(Jf);               //Milnor number of \Sigma(f)
108

> ideal fj=f,Jf;
> fj;
fj[1]=xy
fj[2]=x4+x2y23+y37
fj[3]=-4x4+21x2y23+37y37
> matrix M=jacob(fj);
> print(M);
y,           x,
4x3+2xy23,   23x2y22+37y36,
-16x3+42xy23,483x2y22+1369y36
> ideal K=minor(M,2);
> K;
K[1]=4x4-21x2y23-37y37
K[2]=-16x4-441x2y23-1369y37
K[3]=-2300x5y22-6068x3y36-1184xy59
> vdim(std(K));  //number of cusps
147
```

Therefore, $\mu(\Sigma(f)) = 108$ and $\sharp A_2(f) = 147$.

One way to find the ideal of definition of discriminant curve $\Delta(f)$ of f is elimination of variables. In SINGULAR using elimination of variables is as follows:

```
> ring R=0,(x,y,X,Y),ds;
> ideal f=xy,x4+y37+x2y23;
> ideal Jf=4x4-21x2y23-37y37; //jacobian ideal of f
> ideal I=X-f[1],Y-f[2],Jf;I;
I[1]=X-xy
I[2]=Y-x4-x2y23-y37
I[3]=4x4-21x2y23-37y37
> ideal G=eliminate(I,xy); //eliminate x and y variables of I
```

The defining equation of the discriminant of the germ $f(x,y) = (xy, x^4 + y^{37} + x^2y^{23})$ can not be computed due to lack of memory using the SINGULAR.

Another way to find discriminant curve is $\Delta(f) = V(\mathcal{F}_0(f))$. Now let us use PRESMATRIX to compute an HMP-matrix to calculate the ideals $\mathcal{F}_0(f)$ and $\mathcal{F}_1(f)$.

In our usual setting we have that $\mathcal{X} = V(I) = \Sigma(f) \subset \mathbb{C}^2$, with $I = Jf = \langle 4x^4 - 21x^2y^{23} - 37y^{37}\rangle$, $f\colon \Sigma(f) \to \mathbb{C}^2$ be given by

$$(x,y) \mapsto (xy, x^4 + y^{37} + x^2y^{23}).$$

$A = \mathbb{C}[X,Y]_{\langle X,Y\rangle}, \widetilde{A} = \mathbb{C}[X]_{\langle X\rangle}, B = \left(\frac{\mathbb{C}[x,y]}{\langle 4x^4-21x^2y^{23}-37y^{37}\rangle}\right)_{\langle x,y\rangle}$, ϕ is given by $X \mapsto xy$ and $Y \mapsto x^4 + y^{37} + x^2y^{23}$, and the pushforward module B is minimally generated by $\{1, x, x^2, x^3, y, y^2, y^3, y^4, \ldots, y^{37}\}$, as an $\tilde{A}$-module.

```
> LIB "presmatrix.lib";     //library to compute an HMP-matrix
> ring A=0,(X,Y),ds;        //target ring
> ring B=0,(x,y),ds;        //source ring
> ideal f=xy,x4+y37+x2y23;  //map germ f
> map phi=A, f;             //map phi:A-->B
> ideal I=det(jacob(f));    //jacobian ideal
> presmatrix(phi,I);        //compute an HMP-matrix
//Generators = 1,x,x2,x3,y,y2,...,y37

//     PM
//R^h------>R^h------>Ox------>0;  h = 41, R = A
//too big to be written here
//To access the presentation matrix PM, type:  setring RTPr; PM;
```

The presentation matrix Λ for $f_*\mathcal{O}_{\Sigma(f)}$ is a 41×41 matrix, too big to be written here. The total time to obtain such matrix was about 165 seconds.

```
> setring RTPr;  //ring created internally which contain HMP-matrix
> ideal F0=fitting(PM,0);F0; //0th Fitting ideal
                             //definition ideal of discrim. curve
F0[1]=27021145486791125540040525020149332342086133973427710572843 52Y41+
```

```
2881387081492511697823334`0000000000000000000000000000`X130Y9-
```

Using Singular, and $\Delta(f) = V(\mathcal{F}_0(f))$ obtained by algorithm, also we can not obtain $\mu(\Delta(f))$ due to lack of memory. But, by [9]

$$\mu(\Delta(f)) = \mu(\Sigma(f)) + 2(\sharp A_{1,1}(f) + \sharp A_2(f)).$$

If one can calculate the invariants $\sharp A_{1,1}(f) + \sharp A_2(f)$, then we can obtain $\mu(\Delta(f))$ as well. By [9] and [33],

$$\sharp A_{1,1}(f) + \sharp A_2(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^2,0)}}{\mathcal{F}_1(f)}.$$

```
> ideal F1=fitting(PM,1);F1; //1st Fitting ideal
F1; //too big to be written here
> vdim(std(F1));
2886
```

As, $\mu(\Sigma(f)) = 108$ and $\sharp A_{1,1}(f) + \sharp A_2(f) = \dim_{\mathbb{C}} \dfrac{\mathcal{O}_{(\mathbb{C}^2,0)}}{F_1(f)} = 2886$, we have that $\mu(\Delta(f)) = 5880$. In this case, by computation, as $\sharp A_2(f) = 147$ follows that $\sharp A_{1,1}(f) = 2739$.

Finally,

$$\mu(\Sigma(f)) = 108, \quad \mu(\Delta(f)) = 5880, \quad \sharp A_{1,1}(f) = 2739, \quad \sharp A_2(f) = 147.$$

This example shows the importancy of implementation methods in CA-systems to obtain invariants. Next another application of presmatrix algorithm to topological classification in $\mathcal{O}(2,2)$.

4.3.2 Topological classification in $\mathcal{O}(2,2)$

The description of the topological orbits of map germs is a central question in Singularity Theory, even to find when a $\mathcal{K}$-class has a finite (or not) number of topological orbits is in general, an open problem. Concerning complex map germs from the plane to the plane, Gaffney and Mond in [8] described the topological orbits of *semiquasihomogeneous* map germs which have a representative that is finitely determined. In the corank 2 case, there are germs that belong to a given $\mathcal{K}$-class, but are not semiquasihomogenous, in special if the germ belongs to a $\mathcal{K}$-class with a representative $(xy, x^a + y^b)$ with $g.c.d.(a,b) = 1$. The simplest case is the $\mathcal{K}$-class $(xy, x^2 + y^3)$ where the germs $(xy, x^2 + \alpha xy + y^3)$ are not semiquasihomogeneous for any $\alpha \neq 0$, but are $\mathcal{A}$-equivalent to $(xy, x^2 + y^3)$ and there exists only one topological orbit, see [8, Example 5.11].

For corank 2 map germs from $\mathbb{C}^2$ to $\mathbb{C}^2$ Gaffney and Mond in [8] ask the following question:

How many different topological types are contained in a given $\mathcal{K}(xy, x^a + y^b)$-orbit?

The method to answer this question is the study of the cusps and transversal double fold points which appear in the discriminant curve of any generic deformation of the germ. Whitney showed in [46], that any real stable map germ in these dimensions has only a finite number of cusps and double folds as singular points of the discriminant curve, Gaffney and Mond in [9] showed sufficient conditions for finite determinacy in terms of the finiteness of the number of these singularities. Moreover, the constancy of them is a necessary and sufficient condition for the topological triviality in a family, [9, Corollary 1.10].

In [29] Miranda, Saia and Soares, showed that for all pairs (a, b), excluding $(2, 3)$ and $(2, 5)$, there exist a non finite number of distinct topological types in each $\mathcal{K}$-orbit, that is, for $f(x, y) = (xy, \alpha x^a + \beta y^b)$ there is at least one family such that each element in the family is $\mathcal{A}$-finitely determined germ and any two of them are not $C_0 - \mathcal{A}$-equivalent. For $(a, b) = (2, 3)$ there is only one, and for $(a, b) = (2, 5)$ there are two distinct topological orbits.

In order to illustrate the use of the PRESMATRIX algorithm we fix $(a, b) = (3, 4)$. First we consider the following family $f_{p,q}(x, y) = (xy, x^3 + y^4 + pxy^2 + qx^2y)$ and analyse its discriminant curve $\Delta(f_{p,q}(x, y))$.

```
> LIB "presmatrix.lib";                 //library to compute an HMP-matrix
> ring A=0,(p,q,X,Y),ds;                //target ring
> ring B=0,(p,q,x,y),ds;                //source ring
> ideal f=p,q,,xy,x3+y4+pxy2+qx2y;      //map germ f
> map phi=A, f;                         //map phi:A-->B
> ideal I=det(jacob(f));                //jacobian ideal
I;
I[1]=-3x3-qx2y+pxy2+4y4
> presmatrix(phi,I);                    //compute an HMP-matrix
//Generators = 1,x,x2,y,y2,y3,y4

//     PM
//R^h------>R^h------>Ox------>0;  h = 7, R = R

Y,      -2/3qX, 0,       -4/3pX,        0,             0,      -7/3,
-4/3pX2,Y,      -2/3qX,  0,             0,             -7/3X,  0,
1/3qXY, -2/5pX2,Y,       0,             -7/3X2-8/15pY,2/5p2X, -5/3qX,
-5/4qX2,0,      -7/4X,   Y,             -3/4pX,        0,      0,
0,      -7/4X2, 0,       -5/4qX2,       Y,             -3/4pX, 0,
-7/4X3, 0,      0,       0,             -5/4qX2,       Y,      -3/4pX,
0,      0,      -3/10pX2,-7/4X3-3/20pXY,3/10p2X2,      -5/4qX2,Y

//To access the presentation matrix PM, type:  setring RTPr; PM;
> setring RTPr;
```

```
> ideal F0=fitting(PM,0);F0;
F0[1]=6912Y7-197568qX5Y4-52416pqX3Y5+1024q3X3Y5-823543X12-1411788pX10Y-
677082p2X8Y2-74284p3X6Y3+729p4X4Y4+840350q2X10Y+281260pq2X8Y2+
114254p2q2X6Y3-116375q4X8Y2-6400pq4X6Y3+345744p3qX11+155232p4qX9Y-
3888p5qX7Y2-377300p2q3X11-73080p3q3X9Y+108p4q3X7Y2-6912p7X10+
122500pq5X11+9000p2q5X9Y+3456p6q2X10-12500q7X11-432p5q4X10
```

Thus, the presentation matrix $\Lambda(f)$ is given by

$$\begin{pmatrix} Y & -\frac{2}{3}qX & 0 & -\frac{4}{3}pX & 0 & 0 & -\frac{7}{3} \\ -\frac{4}{3}pX^2 & Y & -\frac{2}{3}qX & 0 & 0 & -\frac{7}{3}X & 0 \\ \frac{1}{3}qXY & -\frac{2}{5}pX^2 & Y & 0 & -\frac{7}{3}X^2-\frac{8}{15}pY & \frac{2}{5}p^2X & -\frac{5}{3}qX \\ -\frac{5}{4}qX^2 & 0 & -\frac{7}{4}X & Y & -\frac{3}{4}pX & 0 & 0 \\ 0 & -\frac{7}{4}X^2 & 0 & -\frac{5}{4}qX^2 & Y & -\frac{3}{4}pX & 0 \\ -\frac{7}{4}X^3 & 0 & 0 & 0 & -\frac{5}{4}qX^2 & Y & -\frac{3}{4}pX \\ 0 & 0 & -\frac{3}{10}pX^2 & -\frac{7}{4}X^3-\frac{3}{20}pXY & \frac{3}{10}p^2X^2 & -\frac{5}{4}qX^2 & Y \end{pmatrix}$$

Now, the discriminant of this map is given by the 0-th Fitting ideal, and we have:

$$\begin{aligned} I(\Delta(f_{p,q})) = \; & 823543\,X^{12} + (12500\,q^7 - 122500\,pq^5 + 377300\,p^2q^3 - 345744\,p^3q)X^{11} \\ & +(1411788\,p - 840350\,q^2)X^{10}Y + (432\,p^5q^4 - 3456\,p^6q^2 + 6912\,p^7)X^{10} \\ & +(73080\,p^3q^3 - 9000\,p^2q^5 - 155232\,p^4q)X^9Y + (116375\,q^4 - 281260\,pq^2 \\ & +677082\,p^2)X^8Y^2 + (-108\,p^4q^3 + 3888\,p^5q)X^7Y^2 + (6400\,pq^4 - 114254\,p^2q^2 + \\ & 74284\,p^3)X^6Y^3 + 197568\,qX^5Y^4 - 729\,p^4X^4Y^4 + (52416\,pq - 1024\,q^3)X^3Y^5 \\ & -6912\,Y^7 \end{aligned}$$

When $p=\frac{q^2}{4}$, the discriminant curve is Newton degenerate, and $\mu(\Delta(f_{\frac{q^2}{4},q}))=54$, [29].

The next step is to fix values for p,q with $p=\frac{q^2}{4}$ and add monomials of degree 4 in the second entry of f. Consider the new family $f_{p,q,u,v,w}(x,y)=(xy,x^3+y^4+qx^2y+pxy^2+uxy^3+vx^3y+wx^4)$, with $p=1, q=2$. We will not show the presentation matrix and the ideal $\mathcal{F}_0$ here because they are very large.

Analysing $\Delta(f_{\frac{q^2}{4},q,u,v,w})$, $q=2$, we conclude that:

Case 1: if $u\neq v-2w+2$, then the Milnor number of discriminant is 54.

Case 2: if $u=v-2w+2$, then the family is not $\mathcal{A}-$finitely determined.

We set $u=2, v=w=0$ and consider the new family

$$f^s(x,y)=(xy,x^3+y^4+2x^2y+xy^2+2xy^3+x^sy^{s+1})$$

with $s>3$. Now f^s is $\mathcal{K}$-equivalent to (xy,x^3+y^4) for all s. Using PRESMATRIX for various fixed values of s, we obtain the following presentation matrix for $f^s_*\mathcal{O}_{\Sigma(f^s)}$:

$$
\begin{bmatrix}
Y & -\frac{4}{3}\,X & 0 & \frac{16}{9}\,\delta & -\frac{10}{3}\,X & 0 & -\frac{7}{3} \\
\frac{16}{9}\,X\delta & Y & -\frac{4}{3}\,X & -\frac{10}{3}\,X^2 & 0 & -\frac{7}{3}\,X & 0 \\
-\frac{10}{3}\,X^3-\frac{37}{30}\,XY & -\frac{4}{3}\,X^{s+1}+\frac{6}{5}\,X^2 & Y & -\frac{8}{5}\,X\delta & 0 & 0 & \frac{11}{10}\,X \\
-\frac{5}{2}\,X^2 & 0 & -\frac{7}{4}\,X & Y & \delta & -X & 0 \\
0 & -\frac{7}{4}\,X^2 & 0 & -\frac{5}{2}\,X^2 & Y & \delta & -X \\
-\frac{37}{12}\,X^3 & 0 & -\frac{4}{3}\,X^2 & \frac{1}{3}\,XY & -\frac{5}{2}\,X^2 & Y+\frac{2}{3}\,X^2 & \delta \\
\frac{15}{8}\,X^3+\frac{27}{160}\,XY & \beta & X^2(\frac{21}{16}-X^{s-1}) & \alpha & X\left(\frac{9}{16}\,X^s+Y\right) & 0 & Y-\frac{63}{160}\,X
\end{bmatrix}
$$

where $\alpha = -\frac{1}{2}\,XY - \frac{9}{4}\,X^3 - \frac{9}{40}\,X^2 - \frac{1}{40}\,X^s\,(9\,X - 10\,Y)$, $\beta = -\frac{5}{2}\,X^3 - \frac{9}{40}\,X^2 - \frac{3}{4}\,XY$ and $\delta = -\frac{3}{4}X^s - \frac{3}{4}X$.

Computing the Fitting ideals we obtain the following topological invariants:

$$\mu(\Delta(f^s)) = 2s+50, \quad d(f^s) = s+14, \quad \mu(\Sigma(f^s)) = 4 \quad \text{and} \quad c(f^s) = 9.$$

Therefore, for each s we have a distinct topological type.

4.3.3 Target multiple points, $(\mathbb{C}^3,0) \to (\mathbb{C}^3,0)$

Consider the corank 2 quasihomogeneous map germ $f : (\mathbb{C}^3,0) \longrightarrow (\mathbb{C}^3,0)$, given by

$$(x,y,z) \longmapsto (x, yz, z^6+y^2+xz),$$

with weights $(5,3,1)$. The objective here is calculate the r-stable types of f in the target and compute the total number of isolated singularities.

```
> LIB "presmatrix.lib";              //library to compute an HMP-matrix
> ring A=0,(X,Y,Z),ds;               //target ring
> ring B=0,(x,y,z),ds;               //source ring
> ideal f=x,yz,z6+y2+xz;             //map germ f
> map phi=A, f;                      //map phi:A-->B
> ideal I=det(jacob(f));               //jacobian ideal
I;
I[1]=-2y2+xz+6z6
> presmatrix(phi,I);                   //compute a HMP-matrix
//Generators = 1,y,z,z2,z3,z4,z5,z6
//     PM
//R^h------>R^h------>Ox------>0;  h = 8, R = A

Z,     0,    -3/2X,  0,     0,     0,     0,    -4,
-3/2XY,Z,    0,      0,     0,     0,     -4Y,  0,
0,     -4/3Y,Z,      -5/6X, 0,     0,     0,    0,
-4/3Y2,0,    0,      Z,     -5/6X, 0,     0,    0,
0,     0,    -4/3Y2, 0,     Z,     -5/6X, 0,    0,
0,     0,    0,      -4/3Y2,0,     Z,     -5/6X,0,
0,     0,    0,      0,     -4/3Y2,0,     Z,    -5/6X,
0,     0,    -5/24XZ,5/16X2,0,     -4/3Y2,0,    Z
```

```
//To access the presentation matrix PM, type:  setring RTPr; PM;
> setring RTPr;
> ideal F0=fitting(PM,0);F0;
F0[1]=186624Z8+12500X6Z3-1907712X2Y2Z5-84375X8Y2+4224000X4Y4Z2-3538944Y6Z4-
19660800X2Y8Z+16777216Y12
ideal F1=fitting(PM,1);F1;

F1[1]=20736Z6+3125X6Z-23040X2Y2Z3-32000X4Y4-196608Y6Z2
F1[2]=3125X5Z2+131328XY2Z4-179200X3Y4Z+327680XY8
F1[3]=625X4Z3+6912Y2Z5-1875X6Y2+23040X2Y4Z2-65536Y8Z
F1[4]=375X3Z4-1750X5Y2Z+18432XY4Z3+6400X3Y6
F1[5]=675X2Z5-3775X4Y2Z2+6912Y4Z4+47360X2Y6Z-65536Y10
F1[6]=96768XYZ5+9375X7Y-294400X3Y3Z2+655360XY7Z
F1[7]=15625X7Z+560640X3Y2Z3-448000X5Y4-4718592XY6Z2
F1[8]=234375X8Y+6169600X4Y3Z2-24772608Y5Z4-153354240X2Y7Z+234881024Y11

ideal F2=fitting(PM,2);F2;
F2[1]=625X4Z+6912Y2Z3-6400X2Y4
F2[2]=25X3Z2+256XY4Z
F2[3]=25X2Z3+256Y4Z2
F2[4]=3XZ4+5X3Y2Z
F2[5]=27YZ4-50X2Y3Z
F2[6]=20736Z5+3125X6-57600X2Y2Z2
F2[7]=125X6-768X2Y2Z2
F2[8]=125X5Y-768XY3Z2
F2[9]=125X4Y2-768Y4Z2
F2[10]=35X3Y2Z-128XY6
F2[11]=X2Y2Z2
F2[12]=XY2Z3
F2[13]=42Y3Z3-25X2Y5
F2[14]=X3Y4
F2[15]=X2Y4Z
F2[16]=XY5Z
F2[17]=Y6Z
F2[18]=XY4Z2
F2[19]=Y5Z2
F2[20]=X2Y6
F2[21]=XY7
F2[22]=Y8
> vdim(std(F2));
62
```

Theorem 4.10 ([17], Proposition 4.6)**.** *Let $f : (\mathbb{C}^{n+3}, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ with $(\Sigma(f), 0)$ Gorenstein such that the codimension of $V(\mathcal{F}_2(f))$ equals to 3,* *then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} = \sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3.$$

Here $\Sigma(f)$ is Gorenstein, since it is a hypersurface. As the Fitting ideals do not distinguish ordinary k-multiple points from the other points whose multiplicity is k, $V(\mathcal{F}_2(f)) = A_{(1,2)} \cup A_{(1,1,1)} \cup A_3$. As `vdim(std(F2)) = 62`,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_3}{\mathcal{F}_2(f)} = \sharp A_{(1,2)} + \sharp A_{(1,1,1)} + \sharp A_3 = 62.$$

Theorem 4.11 ([17], Proposition 4.9)**.** *With the same hypothesis as Theorem 4.10 we have,*

$$\sharp A_{(1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))}.$$

The notation $(I : J)$ means the quotient ideal of an ideal I by another ideal J in a ring A or

$$(I : J) := \{a \in A \ ; \ aJ \subset I\}.$$

We have that $V(\mathcal{F}_1(f)) = A_2(f) \cup A_{1,1}(f)$, thus to calculate the number of ordinary triple points $A_{(1,1,1)}$, first we need to obtain the ideal $I(A_{1,1}(f))$.

```
> ring r=0,(x,y,z),ds;
> ideal f=x,yz,z6+y2+xz;  ideal jf=det(jacob(f)); ideal fj=f,jf;
> matrix N=jacob(fj);
> ideal K=std(minor(N,3));K;
K[1]=xy+16yz5
K[2]=4y2+xz+36z6
K[3]=xz+16z6
```

Using iterated jacobian ideals, we obtain the ideal K that defines points of type A_2 in the source of f. Now by `eliminate` command we can obtain the ideal $I(A_2(f))$ in the target.

```
> ring R=0,(x,y,z,X,Y,Z),ds;
> ideal K=imap(r,K); ideal f=imap(r,f);
> ideal c=X-f[1],Y-f[2],Z-f[3], K;
> ideal A2=eliminate(c,xyz);A2;
A2[1]=5X2Z+1024Y4
A2[2]=4Z3-25X2Y2
A2[3]=125X4Y+4096Y3Z2
```

Obtaining $I(A_{1,1})$:

```
> setring RTPr;
> ideal A2=imap(R,A2);A2;
A2[1]=5X2Z+1024Y4
A2[2]=4Z3-25X2Y2
A2[3]=125X4Y+4096Y3Z2

> ideal A11=quotient(F1,A2); A11;
A11[1]=135XZ4-130X3Y2Z+256XY6
A11[2]=13824Z5-3125X6+53760X2Y2Z2-131072Y6Z
A11[3]=3125X5Z-41472XY2Z3-12800X3Y4
A11[4]=3125X4Z2-6912Y2Z4-46080X2Y4Z+65536Y8
A11[5]=250X3Z3-125X5Y2-1536XY4Z2
A11[6]=15625X7-335360X3Y2Z2+786432XY6Z

> ideal A111=quotient(A11*A11,F0);A111;
A111[1]=X3
A111[2]=X2Z
A111[3]=Y2Z
A111[4]=XZ2
A111[5]=54Z3-25X2Y2
A111[6]=X2Y2
A111[7]=Y4

> vdim(std(A111));
16
```

Therefore, by Theorem 4.11 and `vdim(std(A111)) = 16`,

$$\sharp A_{(1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{(I(A_{1,1})^2 : \mathcal{F}_0(f))} = 16.$$

Then, $\sharp A_{(1,2)} + \sharp A_3 = 62 - 16 = 46$.

For the quasihomogeneous case, using topological approach, [Ohmoto, [36]], presents formulae to compute these 0-stable singularities from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^3, 0)$. By Torus´s formulas, we obtain

$$\sharp A_3 = 10; \quad \sharp A_{1,1,1} = 16; \quad \sharp A_{1,2} = 36,$$

which can be confirmed directly using the previous presentation matrix.

Remark 4.12. We can do primary decomposition of ideal $\mathcal{F}_1(f)$,

```
> list L=primdecSY(F1);
> L;
[1]:
   [1]:
      _[1]=135Z4-130X2Y2Z+256Y6
      _[2]=-3125X4Z+41472Y2Z3+12800X2Y4
```

```
      _[3]=-250X2Z3+125X4Y2+1536Y4Z2
      _[4]=-82944Z5+3125X6+12800X2Y2Z2
   [2]:................................................................
      _[1]=135Z4-130X2Y2Z+256Y6                                       .
      _[2]=-3125X4Z+41472Y2Z3+12800X2Y4                               .
      _[3]=-250X2Z3+125X4Y2+1536Y4Z2                                  .
      _[4]=-82944Z5+3125X6+12800X2Y2Z2                                .
[2]:                                                                  .
   [1]:                                                               .
      _[1]=5X2Z+1024Y4                                                .
      _[2]=-4Z3+25X2Y2                                                .
      _[3]=125X4+4096Y2Z2                                            A11
   [2]:...............................................                .
      _[1]=5X2Z+1024Y4                               .                .
      _[2]=-4Z3+25X2Y2                               .                .
      _[3]=125X4+4096Y2Z2                            .                .
[3]:                                                 .                .
   [1]:                                              .                .
      _[1]=X                                         .                .
      _[2]=-27Z4+256Y6                               .                .
   [2]:................................................................
      _[1]=X                                         .
      _[2]=-27Z4+256Y6                               .
[4]:                                                A2
   [1]:                                              .
      _[1]=Z                                         .
      _[2]=Y                                         .
   [2]:...............................................
      _[1]=Z
      _[2]=Y
```

Note that the components `L[1][2]` and `L[3][2]` define $I(A_{1,1}(f))$, `L[2][2]` and `L[4][2]` define $I(A_2(f))$.

```
> primdecSY(A2);
[1]:
   [1]:
      _[1]=Z
      _[2]=Y
   [2]:
      _[1]=Z
      _[2]=Y
[2]:
   [1]:
      _[1]=5X2Z+1024Y4
      _[2]=-4Z3+25X2Y2
```

```
      _[3]=125X4+4096Y2Z2
   [2]:
      _[1]=5X2Z+1024Y4
      _[2]=-4Z3+25X2Y2
      _[3]=125X4+4096Y2Z2

> primdecSY(A11);
[1]:
   [1]:
      _[1]=X
      _[2]=-27Z4+256Y6
   [2]:
      _[1]=X
      _[2]=-27Z4+256Y6
[2]:
   [1]:
      _[1]=135Z4-130X2Y2Z+256Y6
      _[2]=-3125X4Z+41472Y2Z3+12800X2Y4
      _[3]=-250X2Z3+125X4Y2+1536Y4Z2
      _[4]=-82944Z5+3125X6+12800X2Y2Z2
   [2]:
      _[1]=135Z4-130X2Y2Z+256Y6
      _[2]=-3125X4Z+41472Y2Z3+12800X2Y4
      _[3]=-250X2Z3+125X4Y2+1536Y4Z2
      _[4]=-82944Z5+3125X6+12800X2Y2Z2
```

4.3.4 Multiple points, $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$

Let $f : (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ be a germ of a finite and generically-one-to-one map. Following [32], the (lifted) double points space of f is the space $D^2(f)$ given by the ideal

$$I^2(f) = (f \times f)^* I_{n+1} + R(\alpha),$$

where I_{n+1} is the ideal defining the diagonal of $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, and $R(\alpha)$ is given by the minors of any matrix α, whose entries $\alpha_{ij} \in \mathcal{O}_{2n}$ satisfy

$$f_j(x) - f_j(x') = \sum_{i=1}^{n} \alpha_{ij}(x,x')(x - x'),$$

for all $1 \leq j \leq n+1$. In [19] it is shown that $D^2(f)$ is a Cohen-Macaulay space of dimension $n-1$ (an extension of this result for map germs from $(\mathbb{C}^n,0)$ to $(\mathbb{C}^p,0)$, with $n \leq p$, can be found in [34]). Set theoretically, $D^2(f)$ is given by the pairs $(x,x') \in \mathbb{C}^n \times \mathbb{C}^n$, such that $f(x) = f(x')$ and, if $x = x'$, then f is singular at x. In [24] the source double points space $D(f) \subset \mathbb{C}^n$ is defined as the image of $D^2(f)$ by the projection on the first component $\pi : (\mathbb{C}^n \times \mathbb{C}^n,0) \to (\mathbb{C}^n,0)$, that is:

$$D(f) = V(\mathcal{F}_0(\pi|_{D^2(f)})).$$

The set $D(f)$ plays an important role. For instance, for map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$, it characterizes finite determinancy. More precisely: a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is finitely $\mathcal{A}$-determined if and only if the Milnor number $\mu(D(f))$ is finite [24, 25].

Computing $D(f)$ for a map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$ can be quite involved but, in the case $p = n + 1$, since $\pi\colon D^2(f) \to \mathbb{C}^n$ is a map from a Cohen-Macaulay space of dimension $n - 1$ to $\mathbb{C}^n$, we can use PRESMATRIX algorithm to do so.

A simple but important application of this algorithm is to compute the multiple spaces in the target $M_k(f)$ of a finite map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. As mentioned before, $M_k(f)$ is the zero set of the ideal $\mathcal{F}_{k-1}(f)$ in $\mathcal{O}_{n+1}$.

Example 4.13. Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a corank 2 map germ given by

$$(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy).$$

```
> ring A=0,(X1,X2,Y),ds;           //target ring
> ring B=0,(x,y),ds;               //source ring
> ideal f=x2,y2,x3+y3+xy;          //map germ f
> map phi=A, f; ideal I=0; //If f: X=C^n-->C^{n+1}, set I=0 (zero ideal)
> presmatrix(phi,I);              //compute a HMP-matrix
//Generators = 1,x,y,xy

//     PM
//R^h------>R^h------>Ox------>0;  h = 4, R = A

Y,     -X1,  -X2,  -1,
-X1^2, Y,    -X1,  -X2,
-X2^2, -X2,  Y,    -X1,
-X1*X2,-X2^2,-X1^2,Y

//To access the presentation matrix PM, type:  setring RTPr; PM;
```

Thus, the implementation yields the following presentation matrix of $f_*\mathcal{O}_2$:

$$\Lambda_f = \begin{bmatrix} Y & -X_1 & -X_2 & -1 \\ -X_1^2 & Y & -X_1 & -X_2 \\ -X_2^2 & -X_2 & Y & -X_1 \\ -X_1X_2 & -X_2^2 & -X_1^2 & Y \end{bmatrix}$$

and we obtain the following Fitting ideals (see Figure 17):

$F_0(f) = \langle X_1^2X_2^2 - 2X_1X_2Y^2 + Y^4 - 2X_1^4X_2 - 2X_1X_2^4 - 8X_1^2X_2^2Y - 2X_1^3Y^2 - 2X_2^3Y^2 + X_1^6 - 2X_1^3X_2^3 + X_2^6\rangle$, whose zeros define the image of f.

$F_1(f) = \langle X_2^2 + X_1Y, Y + X_1X_2, -X_2 + X_1^2\rangle \cap \langle X_1 + X_2 - Y, X_2^2 - X_2Y + Y^2\rangle \cap \langle X_2 + Y, X_1 + Y\rangle \cap \langle -X_1 + X_2^2, Y + X_1X_2, X_1^2 + X_2Y\rangle$. This ideal defines the double points space of f in the target.

$F_2(f) = \langle X_1, X_2, Y\rangle$, which defines the triple points in the image of f, and this indicates that f has exactly one ordinary triple point collapsed in the origin in the target.

The set of double points $D^2(f)$ is given by the ideal
$I^2(f) = \langle (x+u)(y+v), (x+u)(2y^2+2yv+2v^2+x+u), (2x^2+2xu+2u^2+y+v)(y+v), x^2-u^2, y^2-v^2, x^3+y^3+xy-u^3-v^3-uv\rangle$.

Take the projection $\pi : (\mathbb{C}^2\times\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ given by $\pi(x,y,u,v) = (x,y)$. A basis of the vector space $\frac{\mathcal{O}_{D^2(f)}}{\pi^*\mathfrak{m}}$ is given by $\{1, y, u, v, v^2, v^3\}$. By PRESMATRIX, we find the following presentation matrix for $\pi_*\mathcal{O}_{D^2(f)}$:

```
> ring A=0,(x,y),ds;                //target ring
> ring B=0,(x,y,u,v),ds;            //source ring
> ideal pi= x,y;                    //projection map germ pi(x,y,u,v)=(x,y)
> map phi=A, pi;
> ideal I=(x+u)(y+v),(x+u)(2y2+2yv+2v2+x+u),(2x2+2xu+2u2+y+v)(y+v),
          x2-u2,y2-v2,x3+y3+xy-u3-v3-uv                        //I=I(D^2(f))
> presmatrix(phi,I);               //compute a HMP-matrix
//Generators = 1,y,u,v,v2,v3

//     PM
//R^h------>R^h------>Ox------>0;  h = 6, R = A

y,    -1,    0,           0,     0,            0,
0,    y,     0,           0,     -1,           0,
0,    2x,    y-2x2-xy2,   x-x2y, -2x2,         -2,
0,    x2,    0,           y+x2,  1,            0,
0,    0,     0,           x2y,   y+x2,         1,
1/2x2,-1/2x3,1/2x+1/2x2y,-1/2xy,-1/2x+1/2x2y,y+1/2x2

//To access the presentation matrix PM, type:  setring RTPr; PM;

> setring RTPr;
> ideal F1=fitting(PM,1);F1;
F1[1]=x2
F1[2]=xy
F1[3]=y2
> vdim(std(F1));
3
```

$$\Lambda_\pi = \begin{bmatrix} Y & -1 & 0 & 0 & 0 & 0 \\ 0 & Y & 0 & 0 & -1 & 0 \\ 2XY+X^3 & 0 & Y-X^2 & X-X^2Y & -X^2 & -2 \\ 0 & X^2 & 0 & Y+X^2 & 1 & 0 \\ 0 & X^2Y & 0 & X^2Y & Y & 1 \\ \frac{1}{2}X^2 & -\frac{1}{2}X^3 & \frac{1}{2}X+\frac{1}{2}X^2Y & -\frac{1}{2}XY & -\frac{1}{2}X+\frac{1}{2}X^2Y & Y+\frac{1}{2}X^2 \end{bmatrix}$$

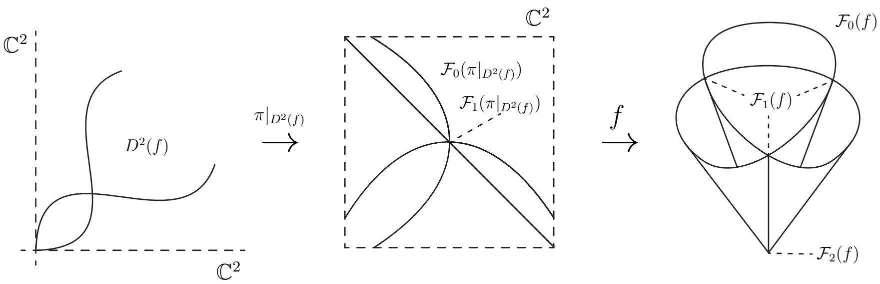


Figure 17: Lifted double points, double points and multiple points in the target.

Note that target of π is the source of f, and then we obtain the following (Figure 17):

1. The ideal $\mathcal{F}_0(\pi|_{D^2(f)}) = \langle (x^2 - xy + y^2)(x+y)(x+y^2)(y+x^2) \rangle$, which defines the source double points space $D(f)$.

2. The ideal $\mathcal{F}_1(\pi|_{D^2(f)}) = \langle x^2, xy, y^2 \rangle$. We may regard the double points of $\pi_{D^2(f)}$ as triple points of f. The codimension of $\mathcal{F}_1(\pi|_{D^2(f)})$ is 3, corresponding to the number of source points in an ordinary triple point, which here have collapsed at 0.

Remark 4.14. We can give two different analytic structures defining the kth source multiple point space of a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. In the first one, we regard the kth source multiple point space as $\pi(D^k(f))$, where $D^k(f) \subseteq (\mathbb{C}^n)^k$ is the lifted kth multiple point space, and π is the projection on the first copy of $\mathbb{C}^n$. Thus, the defining ideal is $\mathcal{F}_0(\pi_{D^k(f)})$. The second structure is given by defining the source multiple points as the preimage by f of $M_k(f)$. It is an open problem to decide whether or not these analytic structures coincide. In the previous example, we computed $\mathcal{F}_0(\pi|_{D^2(f)}) = \langle (x^3+y^3)(x+y^2)(y+x^2) \rangle$, which is precisely the preimage by f of the ideal $\mathcal{F}_1(f)$ computed in Example 4.13.

```
> ring A=0,(X1,X2,Y),ds;           //target ring
> ring B=0,(x,y),ds;               //source ring
> ideal f=x2,y2,x3+y3+xy;          //map germ f
> map phi=A, f; ideal I=0; //If f: X=C^n-->C^{n+1}, set I=0 (zero ideal)
> presmatrix(phi,I);              //compute a HMP-matrix
//Generators = 1,x,y,xy
//     PM
//R^h------>R^h------>Ox------>0;  h = 4, R = A

Y,     -X1,  -X2,  -1,
-X1^2, Y,    -X1,  -X2,
-X2^2, -X2,  Y,    -X1,
-X1*X2,-X2^2,-X1^2,Y
//To access the presentation matrix PM, type:  setring RTPr; PM;
```

```
> setring RTPr;
> ideal F1=fitting(PM,1);F1;
F1[1]=X1*X2-Y^2-X1^3-X2^3-2*X1*X2*Y
F1[2]=X1^2*Y+X2*Y^2+X1^3*X2+X1*X2^2*Y
F1[3]=X2^2*Y+X1*Y^2+X1*X2^3+X1^2*X2*Y
> setring B;
> map G=RTPr,f;
> ideal Df=G(F1);Df;                         //pre-image of F1 by f
Df[1]=-2x4y-2xy4-2x6-4x3y3-2y6-2x5y2-2x2y5
Df[2]=x5y+x2y4+x7+3x4y3+2xy6+2x6y2+3x3y5+y8+x5y4+x2y7
Df[3]=x4y2+xy5+2x6y+3x3y4+y7+x8+3x5y3+2x2y6+x7y2+x4y5
> Df=std(Df);Df;
Df[1]=x4y+xy4+x6+2x3y3+y6+x5y2+x2y5         //pre-image of F1 by f
> factorize(Df[1]);
[1]:
   _[1]=1
   _[2]=x2-xy+y2
   _[3]=x+y
   _[4]=y+x2
   _[5]=x+y2
[2]:
   1,1,1,1,1                      //each factor with power 1
```

Therefore in this exampe, $D(f) = V(\mathcal{F}_0(\pi_{D^k(f)})) = f^{-1}(\mathcal{F}_1(f))$.

4.4 Fast implementation for a presentation of corank 1 map germs in $\mathcal{O}(n, n)$

For any corank 1 finite map germ f in $\mathcal{O}(n, n)$, the set $(\Sigma(f), 0)$ is an $(n-1)$-dimensional Cohen-Macaulay variety, therefore we can use the procedure of Mond-Pellikaan to compute its presentation. Here we show explicitly this construction.

Let $f \in \mathcal{O}(n, n)$ be a finite map germ of corank 1. Choosing linearly adapted coordinates one can write $f(x, z) = (x_1, \dots, x_{n-1}, g(x, z))$, where $x = (x_1, \dots, x_{n-1}) \in \mathbb{C}^{n-1}$, $z \in \mathbb{C}$ and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a polynomial that can be written in the form $g(x, z) = z^{k+1} + h(x, z)$, with $h(x, z) = h_{k-1}(x)z^{k-1} + h_{k-2}(x)z^{k-2} + \cdots h_1(x)z + h_0(x)$ and $h_i(0) = 0$ $h_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $i = 0, \dots, k-1$.

To describe the presentation matrix explicitly we remember that the Jacobian determinant of the matrix of the derivatives of f at any point (x, z) is $J(f) = (k+1)z^k + h_z(x, z)$, where $h_z(x, z)$ denotes the derivative of $h(x, z)$ with respect to the variable z.

For such f it follows that the local algebra $\frac{\mathcal{O}_{(\Sigma(f),0)}}{f^*\mathfrak{m}_{\mathbb{C}^n,0}}$ is isomorphic to $\frac{\mathbb{C}[z]}{\langle z^k \rangle}$, or in other words, $\mathcal{O}_{(\Sigma(f),0)}$ is minimally generated as $\mathcal{O}_n$-module (target) via $f|^*_{(\Sigma(f),0)}$ and in this case a system of generators $\{g_0, g_1, \dots, g_{k-1}\}$ is given by $\{1, z, z^2, \dots, z^{k-1}\}$. The main difficulty to obtain the matrix λ as in (4.3) is to find k relations among the target variables $X_i = f_i$, $i = \{1, \dots, n\}$ and the set of generators $\{1, z, z^2, \dots, z^{k-1}\}$, module the jacobian ideal

$J(f)$. These relations are shown below. As f is of corank 1, we preserve the name of variable x_i, $i = \{1, \dots, n\}$ in the source and in the target.

For the first relation, one has

$$g(x,z) \cdot 1 = z^{k+1} + h_{k-1}(x_1, \dots, x_{n-1}) \cdot z^{k-1} + \cdots + h_1(x_1, \dots, x_{n-1}) \cdot z + h_0(x_1, \dots, x_{n-1}) \cdot 1$$

Let $Y := g(x,z)$, then

$$Y \cdot 1 = z^{k+1} + h_{k-1}(x) \cdot z^{k-1} + \cdots + h_1(x) \cdot z + h_0(x) \cdot 1$$

$$z^{k+1} = (Y - h_0(x)) \cdot 1 - h_1(x) \cdot z - \cdots - h_{k-1}(x) \cdot z^{k-1} \tag{4.4}$$

On the other hand, $(k+1) \cdot z^k + h_z(x,z) = 0$ (in $\mathcal{O}_{(\Sigma(f),0)}$). Then

$$(k+1) \cdot z^k = -(k-1)h_{k-1}(x) \cdot z^{k-2} - \cdots - 2h_2(x) \cdot z - h_1(x) \cdot 1$$

$$z^k = -\left(\frac{k-1}{k+1}\right) h_{k-1}(x) \cdot z^{k-2} - \cdots - \left(\frac{2}{k+1}\right) h_2(x) \cdot z - \left(\frac{1}{k+1}\right) h_1(x) \cdot 1 \tag{4.5}$$

From equality (4.4) and equality (4.5) multiplied by z, results

$$(Y - h_0(x)) \cdot 1 - \left(\frac{k}{k+1}\right) h_1(x) \cdot z - \left(\frac{k-1}{k+1}\right) h_2(x).z^2 - \cdots - \left(\frac{2}{k+1}\right) h_{k-1}(x) \cdot z^{k-1} = 0 \tag{4.6}$$

Now, denote $H_{1,1}(x) = -h_0(x)$, $H_{1,j+1}(x) = -\left(\frac{k+1-j}{k+1}\right) h_j(x)$, $j = 1, \dots, k-1$ and equation (4.6) shows that

$$(Y + H_{1,1}(x)) \cdot g_0 + \sum_{i=1}^{k-1} H_{1,i+1}(x) g_i = 0. \tag{4.7}$$

Therefore the first line of the matrix λ is

$$\begin{bmatrix} Y + H_{1,1}(x) & H_{1,2}(x) & \cdots & H_{1,k}(x) \end{bmatrix}_{1 \times k}.$$

To obtain the second line of the matrix λ, we multiply the equation (4.7) by $g_1 := z$, then

$$Y \cdot g_1 + H_{1,1}(x) \cdot g_1 + \sum_{i=1}^{k-1} H_{1,i+1}(x) g_i \cdot g_1 = 0.$$

As $g_i \cdot g_j = g_{i+j}$, $i + j < k$,

$$Y \cdot g_1 + H_{1,1}(x) \cdot g_1 + H_{1,2}(x) \cdot g_2 + \cdots + H_{1,k-1}(x) \cdot g_{k-1} + H_{1,k}(x) \cdot z^k = 0. \tag{4.8}$$

Substituting the right hand of the equation (4.5) in the equation (4.8), regrouping and renaming the terms

$$H_{2,1}(x) \cdot g_0 + Y \cdot g_1 + H_{2,2}(x) \cdot g_1 + H_{2,3}(x) \cdot g_2 + \cdots + H_{2,k-1}(x) \cdot g_{k-2} + H_{2,k}(x) \cdot g_{k-1} = 0.$$

And the second line of the matrix is given by

$$\left[\begin{array}{cccc} H_{2,1}(x) & Y + H_{2,2}(x) & \cdots & H_{2,k}(x) \end{array} \right]_{1\times k}.$$

Observe that the equation (4.5) does not depend of the variable Y.

Proceeding in this way, to obtain the rth line multiply the $(r-1)th$ line by z or the first line by z^{r-1} and use the equations (4.4) and (4.5). The process is algorithmic and finishes after k lines, where k is the number of generators. To conclude, by [Mond and Pellikaan, [33]] , the relations among the g_i obtained above generate the module Ker ψ in 4.3.

Therefore for any finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $f(x, z) = (x_1, \ldots, x_{n-1}, g(x, z))$, with $g(x, z) = z^{k+1} + h_{k-1}(x)z^{k-1} + h_{k-2}(x)z^{k-2} + \cdots h_1(x)z + h_0(x)$ and $h_i(0) = 0$, $h_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $i = 0, \ldots, k-1$, we have the theorem:

Theorem 4.15 (Theorem 3.1, [30])**.** *The presentation matrix of $\mathcal{O}_{(\Sigma(f),0)}$ over $\mathcal{O}_n$ is given by:*

$$\lambda = \begin{bmatrix} Y + H_{1,1}(x) & H_{1,2}(x) & \cdots & H_{1,k}(x) \\ H_{2,1}(x) & Y + H_{2,2}(x) & & H_{2,k}(x) \\ \vdots & & \ddots & \vdots \\ H_{k,1}(x) & \cdots & & Y + H_{k,k}(x) \end{bmatrix}_{k\times k}$$

where, $H_{i,j} : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ are the polynomials construct above and $(x_1, \ldots, x_{n-1}, Y)$ denote the target variables.

Remark 4.16. From the results of Mond-Pellikaan we can say that the elements of this matrix λ are in the ring $\mathcal{O}_n$, but in this case the entries are polynomials and the determinant of the matrix λ is of the form $Y^k + B$, where B is a polynomial in the variables $x_1, \ldots, x_{n-1}, Y$, with $B(0, \ldots 0, Y) = 0$.

We implemented this procedure to obtain the matrix of Theorem 4.15, called PRESCORANK1. A library SINGULAR of PRESCORANK1, can be downloaded in [Miranda, [28]] or see Subsection 5.3 for a source code. We remark that, with f in its pre-normal form, was possible to obtain a presentation without use Groebner bases.

Remark 4.17. PRESCORANK1 works correctly if f is in pre-normal form. For map germs of corank 1 without being in the pre-normal form, use the more general algorithm PRESMATRIX.

Next some applications by using PRESCORANK1.

4.4.1 Equation of the discriminant

Using PRESCORANK1 we can compute the defining equation of the discriminant in several cases that the classical method called *"Elimination of Variables"* does not finishes because of lack of memory.

For instance, consider the map germ f from $\mathbb{C}^4$ to $\mathbb{C}^4$ given by

$$f(x,y,z,w) = (x,y,z,w^{43} + z^{13}w^3 + y^8w + x^3w^8).$$

The objective is to obtain the ideal $I(\Delta(f))$. The first method is by Fitting ideal, because $\mathcal{F}_0(f) = I(\Delta(f))$. Using PRESCORANK1, the total time to find a 42×42-matrix λ_f presentation matrix and to compute its $\mathcal{F}_0(f)$ Fitting ideal was less than 1 second. The second method is by elimination of variables, see Example 2.7. Using the command `eliminate({W-(f[4]),diff(f[4],w)},{w})` in Maple, where `F[4]=`$w^{43} + z^{13}w^3 + y^8w + x^3w^8$, the total time spent was 970.26 seconds, and in SINGULAR it was not possible to obtain this ideal because lack of memory. We do not show here this $\mathcal{F}_0(f)$ Fitting ideal because its defining equation is very huge.

4.4.2 r-stable types of $f(x,y,z) = (x,y,z^6 + yz + xz^2)$

The determinacy of numerical invariants associated to map germs is a powerful tool in the study of problems of its singularities, in general these invariants appear as schemes associated to the discriminant of stable maps, called stable singularities.

For the particular singularities which are isolated, called 0-stable singularities, the type and also the number of such singularities is very relevant because they hide information about the local geometric behavior of such maps, as we can see in [9], [32], [15], [18]. In general the computation of the 0-stable singularities is not easy, but when the germ is quasihomogeneous, there are several works that show how to compute such numbers in terms of the weights and degrees of quasi homogeneity, se for instance [8], [23], [6], [41].

In the examples λ_f denotes the presentation matrix with respect to $\mathcal{O}_{(\Sigma(f),0)}$.

It is of great interest in Singularity Theory to describe the multiple points spaces (in the target) of any germ f_v given in a deformation of the germ f. We show here how to use this algorithm for the particular case that the deformation is given by $f_v(x,y,z) = (x,y,z^6 + yz + xz^2 + vz^4)$, the same germ consired in section 3.5, with parameter $u = 0$.

This deformation results in beautiful picture for $v < 0$.

Simply we consider the one parameter unfolding F of f, $F : (\mathbb{C} \times \mathbb{C}^3, 0) \to (\mathbb{C} \times \mathbb{C}^3, 0)$ written as

$$F(v,x,y,z) = (v,x,y,z^6 + yz + xz^2 + vz^4)$$

See how to obtain the matrix of a presentation for this map.

```
LIB "prescorank1.lib"; \\library to obtain the presentation matrix
> ring r=0,(v,x,y,z),ds; \\define the ring
> ideal F=v,x,y,z6+yz+xz2+vz4; \\define the map germ
> F;
F[1]=v
F[2]=x
F[3]=y
F[4]=yz+xz2+vz4+z6
```

```
> prescorank1(F);

//     PM
//R^h------>R^h------>Ox------>0;
//h = 5, R = Local target ring with variables:(v,x,y,Y).

Y,            -5/6y,         -2/3x,         0,             -1/3v,
1/18vy,       Y+1/9vx,       -5/6y,         -2/3x+2/9v2,   0,
0,            1/18vy,        Y+1/9vx,       -5/6y,         -2/3x+2/9v2,
1/9xy-1/27v2y,2/9x2-2/27v2x, 1/18vy,        Y+5/9vx-4/27v3,-5/6y,
5/36y2,       7/18xy-1/27v2y,2/9x2-2/27v2x,11/18vy,        Y+5/9vx-4/27v3

//TOTAL TIME = 0 sec

//To access the presentation matrix PM, type:  setring RT; PM;
setring RT;
ideal F0=fitting(PM,0); ideal F1=fitting(PM,1); ideal F2=fitting(PM,2);
> F0;
F0[1]=46656Y5+3125y6+22500xy4Y+43200x2y2Y2+13824x3Y3+32400vy2Y3+62208vxY4+
      256x5y2+2000vx2y4+1024x6Y+10560vx3y2Y-1500v2y4Y+9216vx4Y2-
      6480v2xy2Y2+17280v2x2Y3-13824v3Y4-128v2x4y2-900v3xy4-512v2x5Y-
      4816v3x2y2Y-4352v3x3Y2-192v4y2Y2-9216v4xY3+16v4x3y2+108v5y4+64v4x4Y+
      576v5xy2Y+512v5x2Y2+1024v6Y3
> F1;
F1[1]=1875y4+5400xy2Y+1728x2Y2+1296vY3+256x5+1440vx2y2+1344vx3Y-1080v2y2Y+
      432v2xY2-128v2x4-616v3xy2-592v3x2Y-192v4Y2+16v4x3+72v5y2+64v5xY
F1[2]=1125y3Y+2160xyY2-64x4y-100vxy3+576vx2yY-756v2yY2+32v2x3y+15v3y3-
      344v3xyY-4v4x2y+48v5yY
F1[3]=2700y2Y2+2592xY3+160x3y2+125vy4+384x4Y+180vxy2Y+1728vx2Y2-864v2Y3-
      88v2x2y2-224v2x3Y+44v3y2Y-960v3xY2+12v4xy2+32v4x2Y+128v5Y2
F1[4]=1620yY3-100x2y3-336x3yY+150vy3Y+216vxyY2+65v2xy3+228v2x2yY+48v3yY2-
      9v4y3-32v4xyY
F1[5]=3888Y4+250xy4+1080x2y2Y+576x3Y2+540vy2Y2+3024vxY3+16vx3y2-75v2y4+
      64vx4Y-216v2xy2Y+144v2x2Y2-576v3Y3-4v3x2y2-16v3x3Y-24v4y2Y-64v4xY2
> F2;
F2[1]=96x3+75vy2+72vxY-56v2x2-24v3Y+8v4x
F2[2]=60x2y-45vyY-43v2xy+6v4y
F2[3]=75xy2+54vY2+8vx3-25v2y2+36v2xY-2v3x2-8v4Y
F2[4]=375y3+360xyY+80vx2y+12v2yY-12v3xy
F2[5]=144x2Y+5v2y2-96v2xY+16v4Y
F2[6]=90xyY-4vx2y-33v2yY+v3xy
F2[7]=225y2Y-20vxy2+3v3y2
F2[8]=216xY2+32x4+135vxy2+144vx2Y-8v2x3-33v3y2-32v3xY
F2[9]=540yY2+25vy3+36vxyY+8v3yY
F2[10]=324Y3+20x2y2+48x3Y+15vy2Y+216vxY2-5v2xy2-12v2x2Y-48v3Y2
```

Using the command "prescorank1(F)" we obtain directly the 5×5 matrix λ_F for the presentation

$$\mathcal{O}_3^5 \xrightarrow{\lambda_F} \mathcal{O}_3^5 \xrightarrow{\alpha} \mathcal{O}_{(\Sigma(F),0)} \to 0.$$

Where λ_F is given by

$$\begin{bmatrix} Y & -\frac{5}{6}y & -\frac{2}{3}x & 0 & -\frac{1}{3}v \\ \frac{1}{18}vy & Y+\frac{1}{9}vx & -\frac{5}{6}y & -\frac{2}{3}x+\frac{2}{9}v^2 & 0 \\ 0 & \frac{1}{18}vy & Y+\frac{1}{9}vx & -\frac{5}{6}y & -\frac{2}{3}x+\frac{2}{9}v^2 \\ \frac{1}{9}xy-\frac{1}{27}v^2y & \frac{2}{9}x^2-\frac{2}{27}v^2x & \frac{1}{18}vy & Y+\frac{5}{9}vx-\frac{4}{27}v^3 & -\frac{5}{6}y \\ \frac{5}{36}y^2 & \frac{7}{18}xy-\frac{1}{27}v^2y & \frac{2}{9}x^2-\frac{2}{27}v^2x & \frac{11}{18}vy & Y+\frac{5}{9}vx-\frac{4}{27}v^3 \end{bmatrix}$$

We remark that the time of execution to obtain this matrix was insignificant.

With primary decomposition of these ideals and using for example the Maple to generate the pictures and Cinema 4D or 3DS studio software to model beautiful pictures, we can visualize the multiple points set in the target in real time moving the parameter v. The Figure 18 shows the discriminant and the other stable types of f_v for a fixed real value $v < 0$.

$$\begin{aligned}\Delta(f_v) = \; & V(46656Y^5 + 3125y^6 + 22500xy^4Y + 43200x^2y^2Y^2 + 13824x^3Y^3 + \\ & 32400vy^2Y^3 + 62208vxY^4 + 256x^5y^2 + 2000vx^2y^4 + 1024x^6Y + \\ & 10560vx^3y^2Y - 1500v^2y^4Y + 9216vx^4Y^2 - 6480v^2xy^2Y^2 + \\ & 17280v^2x^2Y^3 - 13824v^3Y^4 - 128v^2x^4y^2 - 900v^3xy^4 - \\ & 512v^2x^5Y - 4816v^3x^2y^2Y - 4352v^3x^3Y^2 - 192v^4y^2Y^2 - \\ & 9216v^4xY^3 + 16v^4x^3y^2 + 108v^5y^4 + 64v^4x^4Y + 576v^5xy^2Y + \\ & 512v^5x^2Y^2 + 1024v^6Y^3).\end{aligned}$$

$$\begin{aligned}A_2(f_v) = \; & V(375y^2 + 1440xY + 32vx^2 - 384v^2Y, \\ & 15525Y^2 + 460x^3 - 750vy^2 - 9090vxY - 271v^2x^2 + 3252v^3Y).\end{aligned}$$

$$\begin{aligned}A_{1,1}(f_v) = \; & V(y, -27Y^2 - 4x^3 - 18vxY + v^2x^2 + 4v^3Y) \cup \\ & V(-25xy^2 - 36x^2Y - 108vY^2 + 5v^2y^2 - 12v^2xY + 4v^4Y, \\ & -40y^2Y^2 - 144xY^3 + 4x^3y^2 + 25vy^4 + 16x^4Y + 132vxy^2Y + 112vx^2Y^2 + \\ & 240v^2Y^3, 125y^4 + 680xy^2Y + 720x^2Y^2 + 1296vY^3 + 4vx^2y^2 + 16vx^3Y - \\ & 8v^2y^2Y - 48v^2xY^2, 108Y^3 - 5x^2y^2 - 20x^3Y - 5vy^2Y + v^2xy^2 + \\ & 4v^2x^2Y - 4v^3Y^2, -25y^2Y - 36xY^2 - 5vxy^2 - 20vx^2Y - 12v^2Y^2 + \\ & v^3y^2 + 4v^3xY, -540Y^2 + 64x^3 + 25vy^2 - 108vxY - 32v^2x^2 + 20v^3Y + 4v^4x, \\ & 125y^2 + 180xY + 64vx^2 - 48v^2Y - 32v^3x + 4v^5, -3024Y^4 - 25xy^4 + 4x^2y^2Y + \\ & 416x^3Y^2 + 132vy^2Y^2 - 288vxY^3 + 5v^2y^4 - 80v^2x^2Y^2 + 160v^3Y^3).\end{aligned}$$

$$A_3(f_v) = \; V(Y, y, x) \cup V(75y^2 + 256xY, -15Y + vx, -5x + 3v^2).$$

$$A_{1,2}(f_v) = \; V(275y^2 + 972xY, -495Y + 8vx, -60x + 11v^2) \cup V(y, 9Y + vx, -3x + v^2).$$

$$A_{1,1,1}(f_v) = \; V(Y, y, -4x + v^2).$$

For $v < 0$ we can see seven real 0-stable singularities.

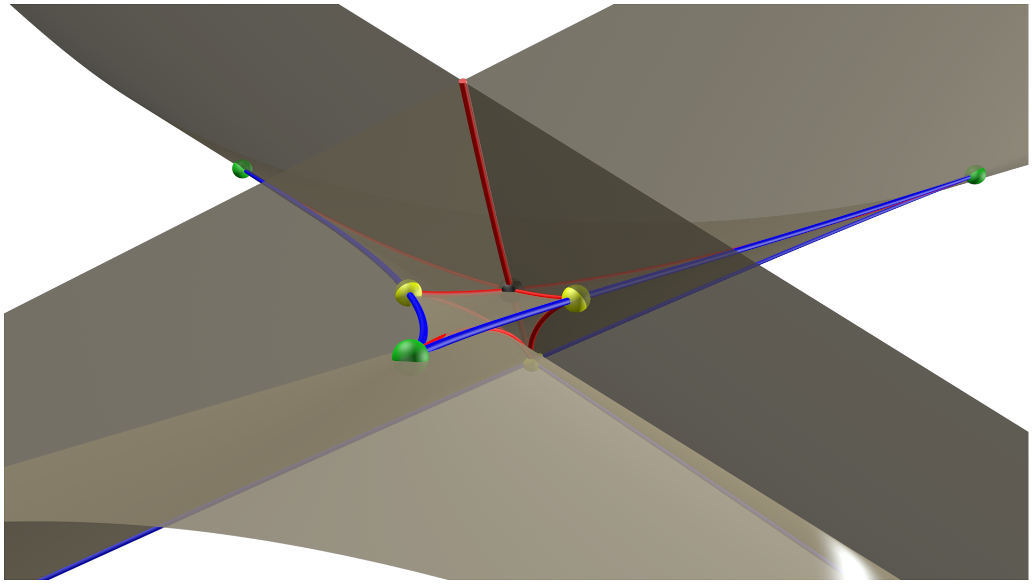


Figure 18: Real part of the discriminant of $f_v(x,y,z) = (x,y,z^6+xz+yz^2+vz^4)$.

4.4.3 Number of points $A_{\mathcal{P}}$, for $|\mathcal{P}|=5$

Example 4.18. Let $f:(\mathbb{C}^5,0)\longmapsto(\mathbb{C}^5,0)$ be given by:

$$f(a,b,c,d,w) = (a,b,c,d,w^{11}+aw^4+bw^3+cw^2+d^3w^2+dw).$$

By [17, Corollary 4.5], we obtain that

$$\sharp A_5 + \sharp A_{(4,1)} + \sharp A_{(3,2)} + \sharp A_{(3,1,1)} + \sharp A_{(2,2,1)} + \sharp A_{(2,1,1,1)} + \sharp A_{(1,1,1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_5}{\mathcal{F}_4(f)} = 252.$$

This germ is not quasihomogeneous then we can not apply the Theorem 3.8 to count each one of the 0-stable singularities. In this case we can apply Theorem 3.9 and IDEALSOURCE implementation in section 3.3 to obtain the ideal $\mathcal{I}^{\ell}(f,\mathcal{P})$. See the number of isolated singularities and time to obtain them in the table below for this example. The time is using the algorithm IDEALSOURCE in Maple.

$A_{\mathcal{P}}$	$\sharp A_{\mathcal{P}}$	time in seconds
A_5	06	0.17
$A_{4,1}$	30	1.43
$A_{3,2}$	30	0.46
$A_{3,1,1}$	60	5.82
$A_{2,2,1}$	60	18.45
$A_{2,1,1,1}$	60	284.63
$A_{1,1,1,1,1}$	06	5595.39

The third column is the time spent (in seconds) to obtain the definition ideal of the $A_{\mathcal{P}}$-singularities in the source, $|\mathcal{P}|=5$.

In the table above the time to compute the ideal that defines $A_{1,1,1,1,1}$ was about 93min.

The time to obtain the presentation matrix to this example using the implementation PRESCORANK1 in Maple was about 0.04sec. The time to obtain the ideal $\mathcal{F}_4(f)$ (`ideal F4=fitting(PM,4)`) and to calculate the $\dim_{\mathbb{C}} \frac{\mathcal{O}_5}{\mathcal{F}_4(f)}$ (`vdim(std(F4))`), was about 27 seconds.

Note that $\sharp A_5 + \sharp A_{(4,1)} + \sharp A_{(3,2)} + \sharp A_{(3,1,1)} + \sharp A_{(2,2,1)} + \sharp A_{(2,1,1,1)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_5}{\mathcal{F}_4(f)} - \sharp A_{(1,1,1,1,1)}$. Then, using together these two algorithms IDEALSOUCE and PRESCORANK1 we can obtain $\sharp A_{(1,1,1,1,1)} = 06$ in about $(0.17 + 1.43 + 0.46 + 5.82 + 18.45 + 284.63) + 27 = 337.96$sec.)

5 Singular–Libraries

5.1 IDEALSOURCE.LIB

Source code in Singular and Maple to compute ideals $\mathcal{I}^{\ell}(f, \mathcal{P})$ that defines multiple point spaces in the source of corank 1 map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. The idealsource.lib file is available in [28].

Source code of IDEALSOURCE in Singular

```
version="$Id: idealsourcelib 2018-01-23";
category="Theory of singularities";
info="
LIBRARY:  idealsource.lib       Comp. source ideals corank1 maps C^n-->C^n
AUTHORS:  Aldicio Jose Miranda,  aldicio@ufu.br,

PROCEDURES:
  idealsource();
";
//----------------------------------------------------------------------
proc idealsource(ideal f, ideal P) // P= partition of n <= dim. source ring
{
 option(noredefine);
 string  RSName=nameof(basering);       // name of source ring
 string varRSName=varstr(basering);
 string varParam=parstr(basering);
 ideal  If=ideal(f);
 int sP=size(P);                        //size of partition P
 int sf=size(f);                        //dim. of source/target
 poly Dim; Dim=0;

 for(int i=1; i<=sP;i++)
   {
     Dim=Dim+P[i];
     if (Dim > sf)
```

```
      {
        ERROR("Use partition for integer <= "+string(sf));
      }
  }
for(int i=1; i<= (sP-1); i++)
  {
    if(P[i+1] < P[i])
     {
      ERROR("Use a non decreasing partition!");
     }
   }
execute("setring "+RSName); //go to source ring
print("//f:(C^"+string(sf)+",0)--->(C^"+string(sf)+",0) ;
           Partition = ["+ string(P)+"]");
string s;   list H;  list L;
int dimF=nvars(basering);       //dimF = dim.of source
if(sf!=dimF)
  {
    ERROR("The dimensions of the source and target are different.");
  }
for(int j=1; j <=sP; j++)    //create zij variables
  {
     for (int i=0;i <= P[j];i++)
       {
        s= string(z)+string(i)+string(j);
        L[i+1]=s;
       }
     H[j]=L;
  }
if(varParam <>"")
  {
   string Nvars= "(0,"+varParam+"),("+string(H)+","+varRSName+"),dp";
  }
    else
      {
        string Nvars= "0,("+string(H)+","+varRSName+"),ds";
      }
poly Iz;Iz=0;
for(int j=1; j<=sf;j++)
  {
    Iz=Iz+var(j);
  }
for (int i=1; i < sf;i++)
  {
    Iz= subst(Iz,If[i],0);
```

```
  }
//-----------------------------NEW RING------------------------------

execute("ring NewR=" +Nvars+";" );
execute("ideal If=imap("+RSName+",If)");
execute("ideal P=imap("+RSName+",P)");
execute("poly Iz=imap("+RSName+",Iz)");
string  NewRName=nameof(basering);
string varParam=parstr(basering);
string s1=string(P);                      //convert ideal P to string
execute("intvec Q = "+s1);                //convert string s1 to intvec Q.
string s2; s2=string(z01);
poly g = If[size(If)]; poly m;  m=0;
int l=size(P); int sf=size(If);
list Var0;                            //string variables z0k.
ideal MP;                             //MP = definition ideal of the
                                      //     multiple points
for (int i=1; i <= l;i++)
  {
    m=m+P[i];
  }
string m1=string(m);  execute("int m2 = "+m1);
int n = m2+l;
matrix M[n][n];
for(int j=1; j <=n; j++)
  {
    for (int i=1;i <= n;i++)
       {
     M[j,i]=var(j)^(i-1);                  //vandermonde denominator
       }
  }
poly detM=det(M);
matrix N[n][n];
int a;
for (int k=2; k<=n;k++)
  {
     N=M;
     for(int i=1; i<=n;i++)
       {
         N[i,k]=(subst(g,Iz,var(i)));        //k-column of N is fixed
       }
       poly detN=det(N);
       poly g2= simplify(detN/detM,1);
       a=1;
       for (int t=1; t<=l; t++)         //l=size(P)
```

```
            {
              if(k==2)
                {
                  Var0[t]=var(a);
                }
                   for(int j=1; j <=Q[t]; j++)
                     {
                       g2 = subst(g2,var(a+j),var(a));  //subs z0k by zik
                     }
                    a=a+(Q[t]+1);
            }
         MP=MP+g2;
  }
 list L;
 for(int i=n+1; i<=(nvars(basering)-1); i++)
  {
    L[i]=var(i);
  }
 if(varParam <>"")
  {
    string Tvars= "(0,"+varParam+"),("+string(Var0)+","+string(L)+"),dp";
  }
    else
      {
     string Tvars= "0,("+string(Var0)+","+string(L)+"),ds";
      }
 execute("ring MR=" +Tvars+";");
 execute("ideal MP=imap("+NewRName+",MP)");
 print("");
 MP;
 exportto(Top,MR);
 exportto(Top,MP);
 print("");
 print("//To access the ideal MP, type:  setring MR; MP; ");
}
```

Source code of IDEALSOURCE in Maple

```
idealsource:=proc(f,part)
local U,s,p,y1,nf,y2,i,j,k,da,dv,t,H,L,T,n,V,A,h,fz,varf,vf,zz,g,su;
global W;
print(´f´=f);
nf:=nops(f); varf:=indets(f); vf:=nops(varf); fz:=f; g:=f[nf];
if nf <> vf then
```

```
   ERROR("Dimensions of source and target are different.");
fi:
for i to (nf-1) do
   fz[nf]:=subs(f[i]=0,fz[nf]):
     if degree(fz[nf]) <=0 then
       ERROR("f is not finite,verify input!");
     fi;
od;
su:=0;
for i to nops(part) do
   su:=part[i]+su;
od:
if su > vf then
  ERROR(`Use partition for integer <= source dimension.`);
fi:
zz:=indets(fz[nf])[]; print(Partition = part);
y1:=0; y2:=0;
for j to nops(part) do
   for k from 0 to part[j] do
      y1:=y1+1; y2:=y2+1:
      H[y1]:=z[j,k];
      t[y2]:=z[j,k+1]=z[j,0];
           if k = part[j] then
             y2:=y2-1;
           fi;
   od;
od;
L:=[seq(H[n], n=1..y1)]:   T:=[seq(t[n], n=1..y2)];
y2:=0:  y1:=0:
s:=0;
V:=vandermonde(L):
for n from 2 to nops(L) do
   dv:=factor(det(V)):
       for i to nops(L) do
          V[i,n]:=subs(zz=V[i,2],g):
       od:
   A:=convert(V,matrix):
   da:=factor(det(A)):
   s:=s+1;
   U[n-1]:=subs(T,simplify(da/dv)):
   print(h[s]=U[n-1]);
   W[s]:=U[n-1]:
od:
end:
```

Meaning of some commands.

seq: the seq command is used to construct a sequence of values. The most typical calling sequence is seq(f(i), i = 1..n) which generates the sequence $f(1), f(2), \ldots, f(n)$.

vandermonde(L): The function vandermonde(L) returns the Vandermonde matrix formed from the elements of the list L. This square matrix has as its $(i, j)^{th}$ entry $L[i]^{(j-1)}$.

subs: substitute subexpressions into an expression. ex.: $subs(x = 2, x^2 + 1) = 5$.

To use this implementation entry with the following command: $idealsource(f, \mathcal{P})$, where $f(x, z) = (x, g(x, z))$. For example: $idealsource([x, y, z^4 + yz + xz^2], [1, 1])$ compute the ideal $\mathcal{I}^2(f, (1, 1))$ that defines $D^2(f, (, 1, 1)) = D^2(f) \subset \mathbb{C}^4$ of map $f(x, y, z) = (x, y, z^4 + yz + xz^2)$.

> idealsource($[x, y, z^4 + yz + xz^2], [1, 1]$)

$$f = [x, y, z^4 + yz + xz^2]$$
$$Partition = [1, 1]$$
$$h_1 = 2\, {z_{1,0}}^2 z_{2,0} + 2\, z_{1,0} {z_{2,0}}^2 + y$$
$$h_2 = -{z_{1,0}}^2 - 4\, z_{1,0} z_{2,0} - {z_{2,0}}^2 + x$$
$$h_3 = 2\, z_{1,0} + 2\, z_{2,0}.$$

5.2 PRESMATRIX.LIB

Source code of PRESMATRIX library in SINGULAR.

```
version="$Id: presmatrix.lib  2013-12-03 $";
category="Singularity Theory, Commutative Algebra";
info="
LIBRARY:  presmatrix.lib         Compute presentation matrix

AUTHORS:  Aldicio Jose Miranda,            aldicio@ufu.br
          Guillermo Penafort-Sanchis,      guille.elrojo@gmail.com

PROCEDURES:
  presmatrix(Pullf, ISource); compute presentation matrix

AUXILIARY PROCEDURES:
  Sols(nJ);          return solutions of linear system of parameters
  Row(rh,D,A,h);     compute rh^th line of the presentation matrix
  PolyGen(d);        create polynomial with coefs = 1 and degree d
 ";
 LIB "matrix.lib";
 LIB "rootsur.lib";
 LIB "ring.lib";
```

```
 LIB "homolog.lib";
//------------------------------------------------------------------------
proc presmatrix(map Pullf, ideal ISource)
{
option(noredefine);
string RSName=nameof(basering);               // name of source ring
execute("int E=ord_test("+RSName+")");
if (E!=-1)
  {
    print("//No local order ring! Get generators using local order");
  }
ideal  IPullf=ideal(Pullf);
execute("setring "+RSName);
string RTName=nameof(preimage(Pullf));       //name of target ring
int    l=nvars(basering);
execute("int n=nvars("+RTName+")-1");
int n2=size(IPullf);
if((n+1)!=n2)
{
ERROR("Target dimension must be equal the number of entries of map!");
}
execute("string Tvars=varstr("+RTName+")");
ideal   IPullf1=IPullf[1..n],ISource;
ideal   KB= kbase(std(IPullf1));             //generators
int     K=size(KB);
if(K==0)
{
ERROR("The number of generators must be greater than zero. Verify input!");
}
matrix  Gen[1][K];
for(int r1=1;r1<=K;r1++)
 {
   Gen[1,r1] = KB[K-r1+1];
 }
print("//Generators = "+string(Gen));
int     h=K;
ring    RG=0,G(1..h),ds;  //auxiliar ring
ring    RT=0,(X(1..n),Y),ds;  //auxiliar target ring
matrix  M[h][h];          //auxiliar matrix
matrix  Pres=diag(Y,h);          //presentation matrix
ring RS=0,(x(1..l)),ds;                  //internal auxiliary ring source
execute("ideal IPf=fetch("+RSName+",IPullf)");     //mapping "Pullf" to RS
execute("ideal I=fetch("+RSName+",ISource)");      //mapping "I" to RS
map Pf=RT,IPf;
int done;      //indicates row done
```

```
for(int rh=1;rh<=h;rh++)
  {
   int d=1;
   done=0;
   while(done==0)
    {
      setring RT;
      matrix  Pols=PolyGen(d);
      int     np=size(Pols[1,1])+(h-1)*size(Pols[1,2])-1;
      poly    D=Pols[1,1];
      poly    A=Pols[1,2];
      ring    RTP1=(0,a(1..np)),(X(1..n),Y),ds;
      poly    D=fetch(RT,D);
      poly    A=fetch(RT,A);
      list    ML=Row(rh,D,A,h);
      matrix  M=ML[1];
      list    ParList=ML[2];
      map     mM=RG,M;
      ring    RSP1=(0,a(1..np)),x(1..l),dp;
      ideal   IPf=fetch(RS,IPf);
      map     Pf=RTP1,IPf;
      ideal   I=fetch(RS,I);  I=std(I);
      execute("matrix Gen=fetch("+RSName+",Gen)");
      poly p=(Pf(mM)[1])*Gen[1,1];
        for(int c=2;c<=h;c++)
          {
            p=p+Pf(mM)[c]*Gen[1,c];
          }
           p=reduce(p,I);
//-----------------------------------------------------------------------
            ideal J;
            poly u=1;
            for(int v=1;v<=l;v++)
            {
              u=x(v)*u;
            }
             matrix C=coef(p,u);
             J=ideal(submat(C,2,1..ncols( C)));
             int sj=size(J); int i=1; int k;
             ring RP=0,(a(1..np)),dp;
             ideal J=imap(RSP1,J);
             list  L=Sols(J);
              if(L[1]==0)
                {
                  d++;
```

```
               }
                 else
               {
            done=1;
            setring RT;
            list  L=imap(RP,L);
            list ParList=imap(RTP1,ParList);
            int   szS=size(L[2]); int   pr;
  for(int i=1;i<=szS;i++)
   {
    pr=L[2][i][1];
    Pres[rh,ParList[pr][2]]=Pres[rh,ParList[pr][2]]+L[2][i][2]*ParList[pr][1];
   }
  }
 }
}
  execute("ring RTPr = 0, ("+Tvars+"),ds");
  matrix PM[h][h]=fetch(RT,Pres);
  print("");
  print("//     PM");
  print("//R^h------>R^h------>Ox------>0;  h = "+string(h)+", R = "+RTName);
  print("");
  print(PM);
  exportto(Top,RTPr);
  exportto(Top,PM);
  print("");
  //print("");
  print("//To access the presentation matrix PM, type:  setring RTPr; PM; ");
}
//-------- proc Sols return solutions of linear system of parameters-----
proc Sols(ideal nJ)
{
  list L;
  nJ=std(nJ);
   if(reduce(1,nJ)==0)
    {
      L[1]=0;  L[2]=list();
    }
      else
       {
         L[1]=1;  L[2]=list();  poly LM, Su, Ji;number LC;
         int s=size(nJ); int  pos=1; int  i=1;
         while(L[1]==1 && i<=s)
          {
            Ji=nJ[i];
```

```
            LC=leadcoef(Ji);
             if(LC!=0)
               {
                 LM=leadmonom(Ji);
                 Su=-Ji/LC+LM;
                    for(int j=i+1;j<=s;j++)
                     {
                       nJ[j]=subst(nJ[j],LM,Su);
                     }
               }
              i++;
             }
               for(int i=s;i>=1;i--)
                 {
                   Ji=nJ[i];
                   if(Ji!=0)
                 {
                  LM=leadmonom(Ji);
                  while(univariate(Ji)==0)
                    {
                      LM=leadmonom(Ji);
                      nJ=subst(nJ,LM,0);
                      Ji=nJ[i];
                    }
                  LM=leadmonom(Ji);
                  Su=-Ji/leadcoef(Ji)+LM;
                  L[2][pos]=list(univariate(Ji),Su);
                  pos++;
                 nJ=subst(nJ,LM,Su);
             }
            }
           }
       return(L);
}
//-------------------------------------------------------------
proc Row(int rh,poly D,poly A, int h)
{
  matrix M[1][h];
  D=D-Y;
  int sD=size(D);
  int sA=size(A);
  list ParList;
  int c=1;
    for(int j=1;j<rh;j++)
      {
```

```
        for(int i=1;i<=sA;i++)
          {
            M[1,j]=A[i]*par( c)+M[1,j];
            ParList[c]=list(A[i],j);
            c++;
          }
      }
      for(int i=1;i<=sD;i++)
       {
        M[1,rh]=D[i]*par( c)+M[1,rh];
        ParList[c]=list(D[i],rh);
        c++;
       }
      M[1,rh]=M[1,rh]+Y;
      for(int j=rh+1;j<=h;j++)
      {
       for(int i=1;i<=sA;i++)
         {
           M[1,j]=A[i]*par( c)+M[1,j];
           ParList[c]=list(A[i],j);
           c++;
         }
      }
  list L;
  L[1]=M;L[2]=ParList;
  return(L);
}
//-------------------------------------------------------------------
proc PolyGen(int d)
 {
    poly D = sparsepoly(0,d,0,1);
    poly PY;
     for(int i=1;i<=d;i++)
       {
        PY=PY+Y^i;
       }
     poly A = D-PY;
     matrix Pols[1][2]=D,A;
     return(Pols);
 }
```

5.3 PRESCORANK1.LIB

Source code of PRESCORANK1 library in SINGULAR.

```
version="$Id: prescorank1.lib  2016-07-15 $";
category="Singularity Theory";
info="
LIBRARY: prescorank1.lib      //Compute presentation matrix for corank 1
                              //map from C^n to C^n
AUTHORS: Aldicio José Miranda,  aldicio@ufu.br
PROCEDURES:
prescorank1(ideal f)
    ";
//-----------------------------------------------------------------------
proc prescorank1(ideal f)
{
option(noredefine);
string  RSName=nameof(basering); // name of source ring
ideal  If=ideal(f);              //If; ideal that define f
int sf=size(f);
execute("setring "+RSName);   //go to source ring
int    dimF=nvars(basering);  //dimF=dim.of source
if(sf!=dimF)
{
ERROR("The dimensions of the source and target are not equal.");
}
poly J=det(jacob(If));
ideal J1=ideal(J);
if(reduce(1,J1)==0)
  {
    ERROR("No singularity. Verify input!");
  }
if(J==0)
 {
   ERROR("Verify input!");
 }
ideal K=kbase(std(If+J));
int nGer=size(K);
poly Iz=J;poly IzAux=J;
for (int i=1;i< dimF;i++)
  {
    Iz= subst(Iz,var(i),0);
    IzAux= subst(IzAux,var(i),1);
  }
if(size(Iz)> 1)
```

```
 {
  ERROR("The map germ ("+string(f)+") is not in pre-normal form.");
 }
if(deg(Iz)!=deg(IzAux))
 {
  ERROR("The map germ ("+string(f)+") is not in pre-normal form.");
 }
int aux1= size(coeffs(J,var(dimF))[deg(Iz)+1,1]);
if( aux1> 1)
 {
  ERROR("The map germ ("+string(f)+") is not in pre-normal form.");
 }
int d=deg(Iz);
poly coefi = coef(Iz,var(dimF))[2,1];
Iz= -(J-Iz)/coefi;
poly Izz=Iz*var(dimF);
poly t=-If[dimF]+var(dimF)^d-Izz;
matrix Pres[nGer][nGer];
matrix M,N;  int sM;
for (int k=1; k <= (nGer);k++)
  {
   M = coef(t,var(dimF));
   sM = (size(M) div 2);
     if (deg(M[1,1])<nGer)
       {
          for (int j=1;j<=sM;j++)
            {
              Pres[k,deg(M[1,j])+1]=M[2,j];
            }
          t=t*var(dimF);
       }
        else
       {
    t=t-M[1,1]*M[2,1]+M[2,1]*Iz;
    k--;
   }
  }
//------------------------------------------------------------------
string  varY; int truY; list L;
varY="Y"; truY=0;
for(int i=1; i<=dimF-1; i++)
  {
    L[i]=var(i);
  }
    for (int n=1; n<= dimF; n++)
```

```
      {
       for (int j=1; j<= dimF; j++)
         {
           if (varstr(j) == varY)
             {
               truY=1;
               varY="Y"+string(n);
               break;
             }
         }
      if (truY==0)
       {
         break;
       }
}
string Tvars= string(L);
execute("ring RT=0,("+Tvars+","+varY+"),ds;");
matrix PM;
execute("matrix PM[nGer][nGer]=fetch("+RSName+",Pres)");
for(int i=1;i<=nGer;i++)
  {
    PM[i,i]=PM[i,i]+var(dimF);
  }
print("");
print("//     PM");
print("//R^h------>R^h------>Ox------>0;  h = "+string(d)+",
         R = Local target ring with variables:("+Tvars+","+varY+").");
print("");
print(PM);
exportto(Top,RT);
exportto(Top,PM);
print("");
print("//To access the presentation matrix PM, type:  setring RT; PM; ");
}
```

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