# Singularities and Characteristic Classes for Differentiable Maps III

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Boa tarde !　　こんにちは！

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Hoje eu começo com esta fórmula agradável.

今日はこの素敵な公式から始めます.

# Izumiya-Marar formula

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In real $C^\infty$ category, let us consider a $C^\infty$ stable map $f : M^2 \to N^3$ from a surface into $3$-space. View its image singular surface $f(M) \subset N$.

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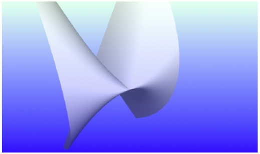
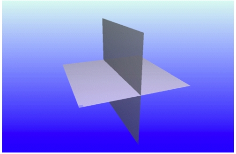
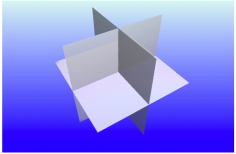
Stable singularities are of the following three types:

$$C := \{\text{Cross Caps}\}, \quad D := \{\text{Double pts}\}, \quad T := \{\text{Triple pts}\}$$







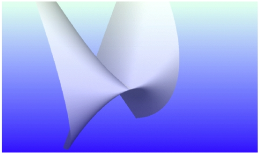
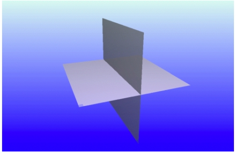
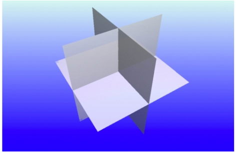
# Izumiya-Marar formula

**The Izumiya-Marar formula** is :

**Theorem 0.1 (Izumiya-Marar)**

*For a $C^\infty$ stable map $M^2 \to N^3$, being $M$ compact, the Euler characteristic of the image singular surface is computed by the following formula:*

$$\chi(f(M)) = \chi(M) + \frac{1}{2}\sharp C + \sharp T$$

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$$M = A_0(f) \cup A_1(f) \longrightarrow f(M) = f(A_0) \cup D \cup T \cup C \ \subset N$$

$$A_0^2(f) := \{ \, x \in A_0(f) \mid \exists \, x' \in A_0(f), \ x' \neq x, \ f(x) = f(x') \, \}$$

$$A_0^3(f) := \left\{ x \in A_0(f) \mid \exists \, x', x'' \in A_0(f), \ s.t. \ \begin{array}{l} x, x', x'' \text{ distint,} \\ \text{having the same image} \end{array} \right\}$$

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Then

$$f_* 1\!\!1_{A_0^2} = 2 1\!\!1_D, \ \ f_* 1\!\!1_{A_0^3} = 3 1\!\!1_T, \ \ f_* 1\!\!1_{A_1} = 1\!\!1_C$$

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$$\begin{array}{rrl} & f_* 1\!\!1_M = & f_* 1\!\!1_{A_0} + 2 1\!\!1_D + 3 1\!\!1_T + 1\!\!1_C \\ -) & 1\!\!1_{f(M)} = & f_* 1\!\!1_{A_0} + 1\!\!1_D + 1\!\!1_T + 1\!\!1_C \\ \hline \end{array}$$

$$f_* 1\!\!1_M - 1\!\!1_{f(M)} = 1\!\!1_D + 2 1\!\!1_T = f_* \left( \tfrac{1}{2} 1\!\!1_{\overline{A_0^2}} + \tfrac{1}{6} 1\!\!1_{A_0^3} - \tfrac{1}{2} 1\!\!1_{A_1} \right)$$

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Take the Integration based on Euler characteristic measure (Nicolas' lecture):

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Take the Integration based on Euler characteristic measure (Nicolas' lecture): By the Fubini theorem,

$$\int_N f_* (\cdot)\, d\chi = \int_M \left( \mathbb{1}_M - \tfrac{1}{2} \mathbb{1}_{\overline{A_0^2}} - \tfrac{1}{6} \mathbb{1}_{A_0^3} + \tfrac{1}{2} \mathbb{1}_{A_1} \right) d\chi$$

Now $\overline{A_0^2}$ is a union of immersed curves, the set of whose intersection are exactly $A_0^3$, so $\chi(\overline{A_0^2}) + \chi(A_0^3) = \chi(\text{disjoint circles}) = 0$, hence

$$\chi(f(M)) = \chi(M) + (\tfrac{1}{2} - \tfrac{1}{6}) \cdot 3\sharp T + \tfrac{1}{2}\sharp C = \chi(M) + \sharp T + \tfrac{1}{2}\sharp C$$

This competes the proof.

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This is valid for complex maps as well. From now on, we work in the complex holomorphic context.

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Apply the CSM class transformation $C_*$ to this equality:

$$C_*(1\!\!1_{f(M)}) = f_* \left( C_*(M) - \frac{1}{2} C_*(\overline{A_0^2}) - \frac{1}{6} C_*(A_0^3) + \frac{1}{2} C_*(A_1) \right)$$

in $H^*(N)$ via the Poincaré dual (omit the notation $\mathrm{Dual}$).

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We call this total cohomology class of $N$
**the Image Chern class of stable maps $f : M \to N$.**

# Izumiya-Marar formula

$$\alpha_{\mathsf{Image}} := 1\!\!1_M - \frac{1}{2}1\!\!1_{\overline{A_0^2}} - \frac{1}{6}1\!\!1_{A_0^3} + \frac{1}{2}1\!\!1_{A_1} \qquad \in \mathcal{F}(M)$$

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$$\chi(f(M)) = \int_N C_*(1\!\!1_{f(M)}) = \int_N C_* f_*(\alpha_{\mathrm{Image}}) = \int_N f_* C_*(\alpha_{\mathrm{Image}})$$

Thus

$$\chi(f(M)) = \int_M C_*(\alpha_{\mathrm{Image}})$$

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- $C_*(\overline{A_0^2}) = [\overline{A_0^2}] + h.o.t = tp(A_0^2) + h.o.t$
  $= (s_0 - c_1) + \left\{ c_1(TM)(s_0 - c_1) + \frac{1}{2}(-s_0^2 - s_1 + 2c_1 s_0 + 2c_2) \right\}$
  Double point formula $+$ higher term

Summing up those classes, we obtain *a universal expression* of **complex Izumiya-Marar formula**:

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**Theorem 0.2 (Ohm)**

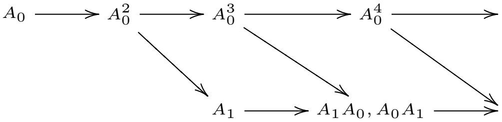
*Given a stable map* $f : M^2 \to N^3$ *of compact complex manifolds. Then it holds that*

$$\chi(f(M)) = \frac{1}{6}\int_M \left( \begin{array}{l} 3c_1(TM)c_1 + 6c_2(TM) - 3c_1(TM)s_0 \\ -c_1^2 - c_2 - c_1 s_0 + s_0^2 + 2s_1 \end{array} \right)$$

*where* $c_i = c_i(f^*TN - TM),\ s_0 = f^*f_*(1),\ s_1 = f^*f_*(c_1)$.

# Image Chern class of maps $M^m \to N^{m+1}$

Universal expression of the image Chern class $C_*(1\!\!1_{f(M)})$ is given in more general form, for a stable complex map $f: M^m \to N^{m+1}$.

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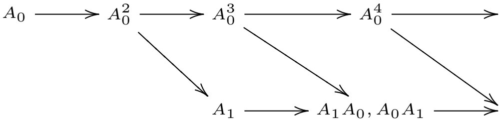


The adjacency relation defines a partial order. By the exclusion-inclusion principle the Möbius inverse formula for $1\!\!1_{\overline{\eta}}$ supported on *the closure of singularities* defines a constructible ft $\alpha_{\mathrm{Image}} \in \mathcal{F}(M)$ s.t.

$$f_*(\alpha_{\mathrm{Image}}) = f_*\left( \begin{array}{l} 1\!\!1_M - \frac{1}{2}1\!\!1_{\overline{A_0^2}} - \frac{1}{6}1\!\!1_{\overline{A_0^3}} + \frac{1}{2}1\!\!1_{\overline{A_1}} \\ \quad -\frac{1}{12}1\!\!1_{\overline{A_0^4}} + \frac{1}{6}1\!\!1_{\overline{A_0A_1}} - \frac{1}{3}1\!\!1_{\overline{A_1A_0}} + \dots \end{array} \right) = 1\!\!1_{f(M)}$$

# Image Chern class of maps $M^m \longrightarrow N^{m+1}$

**Theorem 0.3 (Ohm)**

$\exists$ *universal Segre-SM class* $tp^{SM}(\alpha_{\mathit{Image}})$ *in the difference Chern class* $c_i = c_i(f^*TN - TM)$ *and Landweber-Noviknov class* $s_I = f^* f_*(c^I)$ *s.t.*

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```
SSM := 1 - 1 / 2 tpA00 - 1 / 6 tpA000 + 1 / 2 tpA1 - 1 / 12 tpA0000 + 1 / 6 tpA0A1 - 1 / 3 tpA1A0;
Series[SSM, {t, 0, 3}]
```

$$1 + \frac{1}{2}(c1 - s0)\, t + \frac{1}{6}\left(-c1^2 - c2 - 2\, c1\, s0 + s0^2 + 2\, s1\right) t^2 + \frac{1}{24}\left(2\, c1^3 - 10\, c1\, c2 + 2\, c1^2\, s0 + 2\, c2\, s0 + 3\, c1\, s0^2 - s0^3 + 14\, s01 + 5\, c1\, s1 - 5\, s0\, s1 - 6\, s2\right) t^3 + O[t]^4$$

```
CSM := (1 + c1M t + c2M t^2 + c3M t^3) SSM; Series[CSM, {t, 0, 3}]
```

$$1 + \left(\frac{c1}{2} + c1M - \frac{s0}{2}\right) t + \frac{1}{6}\left(-c1^2 + 3\, c1\, c1M - c2 + 6\, c2M - 2\, c1\, s0 - 3\, c1M\, s0 + s0^2 + 2\, s1\right) t^2 + \frac{1}{24}\left(2\, c1^3 - 4\, c1^2\, c1M - 10\, c1\, c2 - 4\, c1M\, c2 + 12\, c1\, c2M + 24\, c3M + 2\, c1^2\, s0 - 8\, c1\, c1M\, s0 + 2\, c2\, s0 - 12\, c2M\, s0 + 3\, c1\, s0^2 + 4\, c1M\, s0^2 - s0^3 + 14\, s01 + 5\, c1\, s1 + 8\, c1M\, s1 - 5\, s0\, s1 - 6\, s2\right) t^3 + O[t]^4$$

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## Corollary 0.4 (Ohm)

**Izumiya-Marar type formula**: *The Euler characteristic of the image is universally expressed by*

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## Remark 0.5

- Our formula is well structured:
  To compute the image CSM class $C_*(1\!\!1_{f(M)})$, we compute the low dimensional terms of Segre-SM classes $tp^{SM}$ for

$$1\!\!1_{\overline{A_0^2}},\ 1\!\!1_{\overline{A_0^3}},\ 1\!\!1_{\overline{A_0^4}},\ 1\!\!1_{\overline{A_1}},\ 1\!\!1_{\overline{A_0A_1}},\ 1\!\!1_{\overline{A_1A_0}},\cdots$$

  as polynomials in $c_i$ and $s_I$. This can be done by Rimanyi's method + equivariant desingularization method.

# Image Chern class of maps $M^m \to N^{m+1}$

## Remark 0.6

- In exactly the same way, other type image Chern classes such as $C_*(1\!\!1_{\overline{A_0^k(f)}})$ of $k$-th multiple point locus, $C_*(1\!\!1_{\overline{f(A_1(f))}})$ ... etc are also computable in low dimensional terms.

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- Possible to deal with stable maps $M^m \to N^n$ of any $m < n$.
- (Communication with M. Kazarian) There is a nicely behaved closed exponential formula of generating functions of universal SSM classes $tp^{SM}$ for multi-singularities.

# Discriminant Chern class of maps $M^n \longrightarrow N^n$

Let us consider the case of stable complex maps

$$f : M^m \to N^n \ \ (m \geq n)$$

and the *discriminant* (=critical value locus), which is now a reduced hypersurface in $N$

$$D(f) := \overline{f(A_1(f))}$$

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We call $C_*(1\!\!1_{D(f)}) \in H^*(N)$ **the discriminant Chern class of $f$**.

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$$C_*(\alpha_{\text{dis}}) = c(TM) \cdot tp^{SM}(\alpha_{\text{dis}})$$

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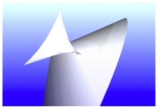
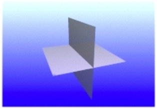
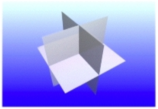
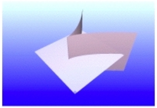
$$\chi(D(f)) = \int_M c(TM) \cdot tp^{SM}(\alpha_{\text{dis}})$$

$tp^{SM}(\alpha_{\text{dis}}) \in H^*(M)$ and $tp^{SM}(1\!\!1_{D(f)}) \in H^*(N)$ up to $n \leq 3$

```
TpA1 := c1 t - c1^2 t^2 + c1^3 t^3;
TpA11 := (- 4 c1^2 - 2 c2 + c1 s1) t^2 +
    (- 4 c1^3 - 10 c1 c2 - 4 c3 + c1 s01 + 3 c1^2 s1 + 2 c2 s1 - c1 s1^2 / 2) t^3;
TpA111 := 1 / 2 (40 c1^3 + 56 c1 c2 + 24 c3 - 2 c1 s01 -
      8 c1^2 s1 - 4 c2 s1 + c1 s1^2 - 4 c1 s2) t^3;
TpA3 :=  (c1 ^ 3 + 3 c1 c2  + 2 c3) t ^ 3;
(* SSM in source *)
SSM := Simplify[ TpA1 - 1 / 2 TpA11 - 1 / 6 TpA111 + 1 / 2 TpA3 ];
Collect[SSM, t]
```

$$c1\,t + \frac{1}{6}\left(6\,c1^2 + 6\,c2 - 3\,c1\,s1\right) t^2 + \frac{1}{6}\left(c1^3 + 11\,c1\,c2 + 6\,c3 - 2\,c1\,s01 - 5\,c1^2\,s1 - 4\,c2\,s1 + c1\,s1^2 + 2\,c1\,s2\right) t^3$$

```
(*discriminant SSM in target *)
```

$$s1\,t + \left(s01 - \frac{s1^2}{2}\right) t^2 + \left(s001 - s01\,s1 + \frac{s1^3}{6} - \frac{s11}{6} + \frac{s3}{6}\right) t^3$$

**Remark 0.7**

The discriminant of stable maps has a particularly nice property: it is a **Free divisor** in the sense of Kyoji Saito.

## Remark 0.7

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It is conjectured by P. Aluffi that under some good condition (e.g., locally quasi-homogeneous), for a free divisor $D$ in non-singular $N$,

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## Remark 0.7

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$$C_*(1\!\!1_{N-D}) = c(\mathrm{Der}(-\log D)) \qquad \in H^*(N)$$

In our setting, $C_*(1\!\!1_{N-D(f)}) = C_*(N) - C_*(D(f)) = c(TN)(1 - tp^{SM}(1\!\!1_{D(f)}))$, hence the conjecture is stated as

$$c(\mathrm{Der}(-\log D(f)) - TN) = 1 - tp^{SM}(1\!\!1_{D(f)})$$

Note that it's universally expressed by the Landweber-Novikov class $s_I$.

Application of ‘ higher Thom polynomials’ $tp^{SM}$ to

**the vanishing topology of weighted homogeneous map-germs.**

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

Consider a finitely determined w. h. map-germ $\eta : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ which is not equiv. to any trivial unfolding of smaller map-germ.

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

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$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\eta} & \mathbb{C}^{n+1} \supset \mathrm{Im}(\eta) \\ i_0 \downarrow & & \downarrow \iota_0 \\ \mathbb{C}^{n+k} & \xrightarrow{F} & \mathbb{C}^{(n+1)+k} \supset \mathrm{Im}(F) \end{array}$$

Take a stable unfolding $F$ of $\eta$ as above: The image hypersurfaces relates as $\mathrm{Im}(\eta) = \iota_0^{-1}(\mathrm{Im}(F))$.

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

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Our interest is to compute the **vanishing Euler characteristics** of the section. i.e.,

$$(-1)^n(\chi(\mathrm{Im}(f_t)) - 1)$$

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

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Our interest is to compute the **vanishing Euler characteristics** of the section. i.e.,

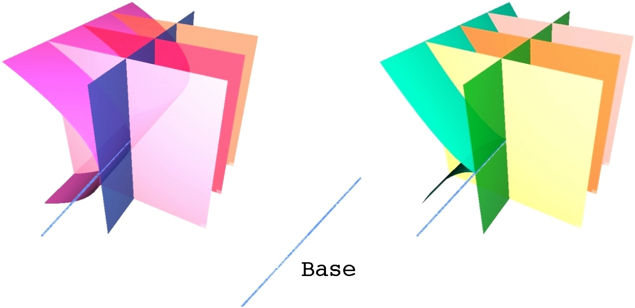
$$(-1)^n(\chi(\mathrm{Im}(f_t)) - 1)$$

If $\eta$ is $\mathcal{A}$-finite, this number is equal to the middle Betti number of the singular Milnor fiber (D. Mond), called Image Milnor number of $\eta$ .

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

As seen before, construct the universal maps over $\mathbb{P}^\infty = B\mathbb{C}^\times$

$$\begin{array}{ccc} E_0(\eta) & \xrightarrow{f_\eta} & E_1(\eta) \supset \mathrm{Im}(f_\eta) \\ i_0 \downarrow & & \downarrow \iota_0 \\ E_0(F) & \xrightarrow{f_F} & E_1(F) \supset \mathrm{Im}(f_F) \end{array}$$



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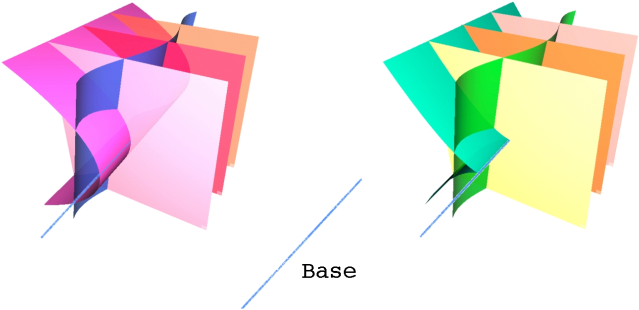

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By the property of our Segre-SM class for transversal pullback,

$$\iota_t^* tp^{SM}(\mathrm{Im}(f_F)) = tp^{SM}(\mathrm{Im}(f_t))$$

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

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By the property of our Segre-SM class for transversal pullback,

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## Theorem 0.8 (Ohm)

**Izumiya-Marar type formula** *holds (Atiyah-Bott localization):*

$$\chi(\mathrm{Im}(f_t)) = \int_{E_0(\eta)} \frac{C_*(\alpha_{\mathit{Image}})}{c_{top}(E_0(\eta))}$$

Image Chern class for stable map $f : M^m \to N^{m+1}$ (again):

$$C_*(1\!\!1_{f(M)}) = f_* C_*(\alpha_{\mathsf{Image}})$$
$$C_*(\alpha_{\mathsf{Image}}) = c(TM) \cdot tp^{SM}(\alpha_{\mathsf{Image}})$$

where the universal SSM class is

$$\begin{aligned} tp^{SM}(\alpha_{\mathsf{Image}}) \; = \; & 1 \\ & + \tfrac{1}{2}(c_1 - s_0) \\ & + \tfrac{1}{2}(s_0^2 + 2s_1 - 2c_1 s_0 - c_1^2 - c_2) \\ & + \tfrac{1}{24}\begin{pmatrix} 2c_1^3 - 10c_1c_2 + 2c_1^2 s_0 + 2c_2 s_0 + 3c_1 s_0^2 \\ -s_0^3 + 14 s_{01} + 5c_1 s_1 - 5 s_0 s_1 - 6 s_2 \end{pmatrix} \\ & + h.o.t \end{aligned}$$

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

For weighted homogeneous map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ with

weights $w_1, w_2$ and degree $d_1, d_2, d_3$. Then,

$$c(f_\eta) = \frac{(1+d_1)(1+d_2)(1+d_3)}{(1+w_1)(1+w_2)},\ s_0 = f_{\eta *}(1) = \frac{d_1 d_2 d_3}{w_1 w_2},\ s_I = f_{\eta *}(c^I) = c^I s_0$$

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

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$$\frac{C_*(\alpha_{\mathsf{Image}})}{c_{top}(E_0(\eta))} = \frac{(1+w_1)(1+w_2) \cdot tp^{SM}(\alpha_{\mathsf{Image}})(f_\eta)}{w_1 w_2}$$

The degree minus $1$ gives the image Milnor number. Let's see examples.

For weighted homogeneous map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^3, 0$,

```
a1 := d1 + d2 + d3 - w1 - w2; a2 := (d1 d2 + (d1 + d2) d3 - (d1 + d2 + d3) w1 + w1^2 - (d1 + d2 + d3 - w1) w2 + w2^2) ;
a3 := (d1 d2 d3 - (d2 d3 + d1 (d2 + d3)) w1 + (d1 + d2 + d3) w1^2 -
     w1^3 - (d1 d2 + (d1 + d2) d3 - (d1 + d2 + d3) w1 + w1^2) w2 + (d1 + d2 + d3 - w1) w2^2 - w2^3);
sa0 := (d1 d2 d3)/(w1 w2); sa1 := sa0 a1;
AAA := (1 + 1/2 (c1 - s0) t + 1/6 (-c1^2 - c2 - 2 c1 s0 + s0^2 + 2 s1) t^2) ((1 + t w1) (1 + t w2))/(w1 w2) /.
   {c1 → a1, c2 → a2, s0 → sa0, s1 → sa1}
-1 + Simplify[Coefficient[AAA, t^2]]
```

```
-1 + 1/(6 w1^3 w2^3) (d1^2 (d2^2 d3^2 - w1^2 w2^2) - w1^2 w2^2 (d2^2 + d3^2 + 5 w1^2 + 3 w1 w2 + 5 w2^2 - 6 d3 (w1 + w2) + 3 d2 (d3 - 2 (w1 + w2))
    3 d1 w1 w2 (w1 w2 (d3 - 2 (w1 + w2)) + d2 (w1 w2 + d3 (w1 + w2))))
```

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

For weighted homogeneous map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^3, 0$,

```
a1 := d1 + d2 + d3 - w1 - w2; a2 := (d1 d2 + (d1 + d2) d3 - (d1 + d2 + d3) w1 + w1^2 - (d1 + d2 + d3 - w1) w2 + w2^2) ;
a3 := (d1 d2 d3 - (d2 d3 + d1 (d2 + d3)) w1 + (d1 + d2 + d3) w1^2 -
    w1^3 - (d1 d2 + (d1 + d2) d3 - (d1 + d2 + d3) w1 + w1^2) w2 + (d1 + d2 + d3 - w1) w2^2 - w2^3);
sa0 := (d1 d2 d3)/(w1 w2); sa1 := sa0 a1;
AAA := (1 + 1/2 (c1 - s0) t + 1/6 (-c1^2 - c2 - 2 c1 s0 + s0^2 + 2 s1) t^2) ((1 + t w1) (1 + t w2))/(w1 w2) /.
  {c1 → a1, c2 → a2, s0 → sa0, s1 → sa1}
-1 + Simplify[Coefficient[AAA, t^2]]
```

$$-1 + \frac{1}{6\, w1^3\, w2^3}\Big(d1^2\,\big(d2^2\, d3^2 - w1^2\, w2^2\big) - w1^2\, w2^2\,\big(d2^2 + d3^2 + 5\, w1^2 + 3\, w1\, w2 + 5\, w2^2 - 6\, d3\, (w1 + w2) + 3\, d2\, (d3 - 2\, (w1 + w2))\big)$$
$$3\, d1\, w1\, w2\, (w1\, w2\, (d3 - 2\, (w1 + w2)) + d2\, (w1\, w2 + d3\, (w1 + w2)))\Big)$$

This coincides with D. Mond's computation (1991): our method is completely different.

# Image Milnor number of map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+1}, 0$

For w. h. germs $\mathbb{C}^3, 0 \to \mathbb{C}^4, 0$ (cf. Mond, Marar, Wik-Atique, Houston...)

```
ImageMilnor[w1_, w2_, w3_, d1_, d2_, d3_, d4_] :=
  1 - 1/(24 w1^4 w2^4 w3^4) (d1^3 (-d2^3 d3^3 d4^3 - 2 d2^2 d3^2 d4^2 w1 w2 w3 + d2 d3 d4 w1^2 w2^2 w3^2 + 2 w1^3 w2^3 w3^3) - 2 d1^2 w1 w2 w3
      (d2^3 d3^2 d4^2 + 2 (d3 + d4) w1^2 w2^2 w3^2 + d2 w1 w2 w3 (-9 d3^2 d4 + 2 w1 w2 w3 + 9 d3 d4 (-d4 + w1 + w2 + w3)) +
        d2^2 d3 d4 (d3^2 d4 - 9 w1 w2 w3 + d3 d4 (d4 - 3 (w1 + w2 + w3)))) - 2 w1^3 w2^3 w3^3 (-d2^3 - d3^3 + 2 d3^2 d4 - d4^3 +
        2 d2^2 (d3 + d4) + d4 w1^2 - 9 d4 w1 w2 + 9 w1^2 w2 + d4 w2^2 + 9 w1 w2^2 - 9 d4 w1 w3 + 9 w1^2 w3 - 9 d4 w2 w3 + 15 w1 w2 w3 +
        9 w2^2 w3 + d4 w3^2 + 9 w1 w3^2 + 9 w2 w3^2 + d3 (2 d4^2 + w1^2 + w2^2 - 9 w2 w3 + w3^2 - 9 w1 (w2 + w3) - 3 d4 (w1 + w2 + w3)) +
        d2 (2 d3^2 + 2 d4^2 + w1^2 - 9 w1 w2 + w2^2 - 9 w1 w3 - 9 w2 w3 + w3^2 - 3 d4 (w1 + w2 + w3) + d3 (9 d4 - 3 (w1 + w2 + w3)))) +
     d1 w1^2 w2^2 w3^2 (d2^3 d3 d4 + 2 d2^2 (9 d3^2 d4 + 9 d3 d4 (d4 - w1 - w2 - w3) - 2 w1 w2 w3) - 2 w1 w2 w3
         (2 d3^2 + 2 d4^2 + w1^2 - 9 w1 w2 + w2^2 - 9 w1 w3 - 9 w2 w3 + w3^2 - 3 d4 (w1 + w2 + w3) + d3 (9 d4 - 3 (w1 + w2 + w3))) +
        d2 (d3^3 d4 + 18 d3^2 d4 (d4 - w1 - w2 - w3) + 6 w1 w2 w3 (-3 d4 + w1 + w2 + w3) +
           d3 (d4^3 - 18 w1 w2 w3 - 18 d4^2 (w1 + w2 + w3) + d4 (17 w1^2 + 17 w2^2 + 6 w2 w3 + 17 w3^2 + 6 w1 (w2 + w3))))));

(* corank one *)
Simplify[ImageMilnor[w1, w2, w0, w1, w2, d1, d2]]

1/(24 w0^4 w1 w2)
(d1 - w0) (d2 - w0) (d1^2 (d2^2 + 3 d2 w0 + 2 w0^2) + d1 w0 (3 d2^2 + 2 w0 (w0 - 2 (w1 + w2)) - d2 (19 w0 + 4 (w1 + w2))) +
   2 w0^2 (d2^2 + d2 (w0 - 2 (w1 + w2)) + 2 (3 w1 w2 + 5 w0 (w1 + w2))))
```

This coincides with all known examples of corank $1$ in $\mathcal{A}$-classification.

For w. h. germs $\mathbb{C}^3, 0 \to \mathbb{C}^4, 0$ (cf. Mond, Marar, Wik-Atique, Houston...)

```
ImageMilnor[w1_, w2_, w3_, d1_, d2_, d3_, d4_] :=
 1 - 1/(24 w1^4 w2^4 w3^4) (d1^3 (-d2^3 d3^3 d4^3 - 2 d2^2 d3^2 d4^2 w1 w2 w3 + d2 d3 d4 w1^2 w2^2 w3^2 + 2 w1^3 w2^3 w3^3) - 2 d1^2 w1 w2 w3
      (d2^3 d3^2 d4^2 + 2 (d3 + d4) w1^2 w2^2 w3^2 + d2 w1 w2 w3 (-9 d3^2 d4 + 2 w1 w2 w3 + 9 d3 d4 (-d4 + w1 + w2 + w3)) +
        d2^2 d3 d4 (d3^2 d4 - 9 w1 w2 w3 + d3 d4 (d4 - 3 (w1 + w2 + w3)))) - 2 w1^3 w2^3 w3^3 (-d2^3 - d3^3 + 2 d3^2 d4 - d4^3 +
        2 d2^2 (d3 + d4) + d4 w1^2 - 9 d4 w1 w2 + 9 w1^2 w2 + d4 w2^2 + 9 w1 w2^2 - 9 d4 w1 w3 + 9 w1^2 w3 - 9 d4 w2 w3 + 15 w1 w2 w3 +
        9 w2^2 w3 + d4 w3^2 + 9 w1 w3^2 + 9 w2 w3^2 + d3 (2 d4^2 + w1^2 + w2^2 - 9 w2 w3 + w3^2 - 9 w1 (w2 + w3) - 3 d4 (w1 + w2 + w3)) +
        d2 (2 d3^2 + 2 d4^2 + w1^2 - 9 w1 w2 + w2^2 - 9 w1 w3 - 9 w2 w3 + w3^2 - 3 d4 (w1 + w2 + w3) + d3 (9 d4 - 3 (w1 + w2 + w3))))
      d1 w1^2 w2^2 w3^2 (d2^3 d3 d4 + 2 d2^2 (9 d3^2 d4 + 9 d3 d4 (d4 - w1 - w2 - w3) - 2 w1 w2 w3) - 2 w1 w2 w3
         (2 d3^2 + 2 d4^2 + w1^2 - 9 w1 w2 + w2^2 - 9 w1 w3 - 9 w2 w3 + w3^2 - 3 d4 (w1 + w2 + w3) + d3 (9 d4 - 3 (w1 + w2 + w3))) +
        d2 (d3^3 d4 + 18 d3^2 d4 (d4 - w1 - w2 - w3) + 6 w1 w2 w3 (-3 d4 + w1 + w2 + w3) +
           d3 (d4^3 - 18 w1 w2 w3 - 18 d4^2 (w1 + w2 + w3) + d4 (17 w1^2 + 17 w2^2 + 6 w2 w3 + 17 w3^2 + 6 w1 (w2 + w3))))));

(* corank one *)
Simplify[ImageMilnor[w1, w2, w0, w1, w2, d1, d2]]
```

```
     1
-----------
24 w0^4 w1 w2
(d1 - w0) (d2 - w0) (d1^2 (d2^2 + 3 d2 w0 + 2 w0^2) + d1 w0 (3 d2^2 + 2 w0 (w0 - 2 (w1 + w2)) - d2 (19 w0 + 4 (w1 + w2))) +
   2 w0^2 (d2^2 + d2 (w0 - 2 (w1 + w2)) + 2 (3 w1 w2 + 5 w0 (w1 + w2))))
```

Notice that our formula is valid for w. h. germs of any corank.

# Discriminant Milnor number of map-germs $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$

For $\mathcal{A}$-finite map-germs $\mathbb{C}^m, 0 \to \mathbb{C}^n, 0 \ \ (m \geq n)$,
the discriminant Milnor number is defined (Damon-Mond):

$$(-1)^{n-1}(\chi(D(f_t)) - 1)$$

In weighted homogeneous case, the discriminant Milnor number can also be obtained by localizing higher Tp:

$$\chi(D(f_t)) = \int_{E_0(\eta)} \frac{C_*(\alpha_{\mathsf{Dis}})}{c_{top}(E_0(\eta))}$$

For example, see the case of $m = n$.

# Discriminant Milnor number of map-germs $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$

Discriminant Chern class for stable map $f : M^n \to N^n$ (again) :

$$C_*(1\!\!1_{f(M)}) = f_* C_*(\alpha_{\mathsf{Dis}})$$
$$C_*(\alpha_{\mathsf{Dis}}) = c(TM) \cdot tp^{SM}(\alpha_{\mathsf{Dis}})$$

where the universal SSM class is

$$\begin{aligned} tp^{SM}(\alpha_{\mathsf{Dis}}) &= c_1 \\ &+\tfrac{1}{6}(6c_1^2 + 6c_2 - 3c_1 s_1) \\ &+\tfrac{1}{6}\left( \begin{array}{l} c_1^3 + 11c_1c_2 + 6c_3 - 2c_1 s_{01} - 5c_1^2 s_1 \\ -4c_2 s_1 + c_1 s_1^2 + 2c_1 s_2 \end{array} \right) \\ &+h.o.t \end{aligned}$$

# Discriminant Milnor number of map-germs $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$

For weighted homogeneous map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$,

```
CCC := 1 + (d1 + d2 - w1 - w2) t + (w1^2 + d1 (d2 - w1 - w2) + w1 w2 + w2^2 - d2 (w1 + w2)) t^2;
aa1 := Coefficient[CCC, t];
aa2 := Coefficient[CCC, t^2];
saa0 := d1 d2  w1^{-1} w2^{-1} ;
saa1 := saa0 aa1; saa2 := saa0 aa1^2;
saa01 := saa0 aa2;
BBB :=
 (s1 t + (s01 - s1^2/2) t^2) (1 + d1 t) (1 + d2 t) d1^{-1} d2^{-1}  /.
  {c1 → aa1, c2 → aa2, s1 → saa1, s01 → saa01, s2 → saa2}
Simplify[Factor[1 - Expand[Coefficient[BBB, t^2]]]]
```

$$\left\{\left\{\frac{(d1\,d2 - 2\,w1\,w2)\left(d1^2 + d2^2 + w1^2 + 2\,d1\,(d2 - w1 - w2) + w2^2 - 2\,d2\,(w1 + w2)\right)}{2\,w1^2\,w2^2}\right\}\right\}$$

```
DisMilnor2[w1_, w2_, d1_, d2_] :=
  (d1 d2 - 2 w1 w2) (d1^2 + d2^2 + w1^2 + 2 d1 (d2 - w1 - w2) + w2^2 - 2 d2 (w1 + w2)) / (2 w1^2 w2^2);
```

This coincides with Gaffney-Mond's computation (1991).

For w. h. germs $\mathbb{C}^3,0 \to \mathbb{C}^3,0$, (cf. Marar-Tari, Saia, Perez, ...)

```
In[15]:= DisMilnor[w1_, w2_, w3_, d1_, d2_, d3_] :=
  -1 + 1/(6 w1^3 w2^3 w3^3) (d1^5 d2^2 d3^2 + 3 d1^4 d2 d3 (d2^2 d3 + d2 d3 (d3 - w1 - w2 - w3) - w1 w2 w3) +
      w1^2 w2^2 w3^2 (d2^3 + d3^3 - 6 w1^3 - 7 w1^2 w2 - 7 w1 w2^2 - 6 w2^3 - 7 w1^2 w3 - 9 w1 w2 w3 - 7 w2^2 w3 - 7 w1 w3^2 - 7 w2 w3^2 - 6 w3^3 -
         8 d3^2 (w1 + w2 + w3) + d3 (13 w1^2 + 13 w2^2 + 15 w2 w3 + 13 w3^2 + 15 w1 (w2 + w3)) + 2 d2^2 (7 d3 - 4 (w1 + w2 + w3)) +
         d2 (14 d3^2 + 13 w1^2 + 13 w2^2 + 15 w2 w3 + 13 w3^2 + 15 w1 (w2 + w3) - 27 d3 (w1 + w2 + w3))) +
      d1^3 (3 d2^4 d3^2 + 6 d2^3 d3^2 (d3 - w1 - w2 - w3) + w1^2 w2^2 w3^2 - 3 d2 d3 w1 w2 w3 (5 d3 - 4 (w1 + w2 + w3)) +
         3 d2^2 d3 (d3^3 - 5 w1 w2 w3 - 2 d3^2 (w1 + w2 + w3) + d3 (w1 + w2 + w3)^2)) +
      d1^2 (d2^5 d3^2 + 3 d2^4 d3^2 (d3 - w1 - w2 - w3) - 2 w1^2 w2^2 w3^2 (-7 d3 + 4 (w1 + w2 + w3)) +
         3 d2^3 d3 (d3^3 - 5 w1 w2 w3 - 2 d3^2 (w1 + w2 + w3) + d3 (w1 + w2 + w3)^2) -
         d2 w1 w2 w3 (15 d3^3 - 14 w1 w2 w3 - 30 d3^2 (w1 + w2 + w3) + 3 d3 (5 w1^2 + 5 w2^2 + 8 w2 w3 + 5 w3^2 + 8 w1 (w2 + w3))) +
         d2^2 d3 (d3^4 - 3 d3^3 (w1 + w2 + w3) + 30 w1 w2 w3 (w1 + w2 + w3) + 3 d3^2 (w1 + w2 + w3)^2 -
            d3 (w1^3 + 3 w1^2 (w2 + w3) + (w2 + w3)^3 + 3 w1 (w2^2 + 14 w2 w3 + w3^2)))) +
      d1 w1 w2 w3 (-3 d2^4 d3 - 3 d2^3 d3 (5 d3 - 4 (w1 + w2 + w3)) + w1 w2 w3 (14 d3^2 + 13 w1^2 + 13 w2^2 +
            15 w2 w3 + 13 w3^2 + 15 w1 (w2 + w3) - 27 d3 (w1 + w2 + w3)) -
         d2^2 (15 d3^3 - 14 w1 w2 w3 - 30 d3^2 (w1 + w2 + w3) + 3 d3 (5 w1^2 + 5 w2^2 + 8 w2 w3 + 5 w3^2 + 8 w1 (w2 + w3))) -
         3 d2 (d3^4 - 4 d3^3 (w1 + w2 + w3) + 9 w1 w2 w3 (w1 + w2 + w3) + d3^2 (5 w1^2 + 5 w2^2 + 8 w2 w3 + 5 w3^2 + 8 w1 (w2 + w3)) -
            d3 (2 w1^3 + 4 w1^2 (w2 + w3) + w1 (4 w2^2 + 21 w2 w3 + 4 w3^2) + 2 (w2^3 + 2 w2^2 w3 + 2 w2 w3^2 + w3^3)))));

  (* corank one *)

In[17]:= Simplify[DisMilnor[w1, w2, w0, w1, w2, d]]
```

Out[17]=

$$\frac{(d - 2\,w0)\left(d^4 - 4\,d^3\,w0 + d^2\,w0\,(8\,w0 - 3\,(w1 + w2)) + 2\,d\,w0^2\,(-4\,w0 + 3\,(w1 + w2)) + 3\,w0^2\left(w0^2 + 2\,w1\,w2 - w0\,(w1 + w2)\right)\right)}{6\,w0^3\,w1\,w2}$$

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  **the Tp theory for $\mathcal{C}$ on $\mathcal{V}$ "=" Intersection theory on $B_{\mathcal{C}}\mathcal{V}$**

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- To local invariants of map-germs, one assigns the **equivariant Segre-SM class** $tp^{SM}$ as "**higher Tp**".
  It has naturality and motivic property.
- If torus action is there, one can **localize the higher Tp** to compute the local invariant (Atiyah-Bott localization).

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- A huge missing part is about Tp for real singularities, in particular

  **Local Vassiliev type inv. "=" Relative Tp.**

A lot of problems are there !

Muito Obrigado !

e ...

Feliz aniversário, Shyuichi !

泉屋先生，誕生日おめでとう！

Mas, não beba muito !