Notes on non-isolated singularities

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Contents

In these lectures we shall give results about non-isolated singularities. We shall omit some proofs. These ones can be find in the litterature. We give many references for the reader.

We shall try to enounce the results as clearly as possible. We hope this will give a lead to students in the maze of the theory of singularity.

1 Critical points of polynomials

1.1 Polynomial functions

Let p_0 be a point of $\mathbb{K}^{n+1}$, where $\mathbb{K}$ is either the field of real numbers $\mathbb{R}$ or the field $\mathbb{C}$ of complex numbers. The point p_0 is called a *critical point* of a $\mathbb{K}$-polynomial function f, if all the partial derivatives $\partial f/\partial X_0, \ldots, \partial f/\partial X_n$ of f vanish at p_0:

$$\partial f/\partial X_0(p_0) = \ldots = \partial f/\partial X_n(p_0) = 0.$$

If the point p_0 is a critical point of f, the value $f(p_0)$ is called a *critical value* of f.

For $\mathbb{K}$-polynomial functions one has a more precise result than Sard's Lemma (see [15] §2 and 3):

Proposition 1.1 *A $\mathbb{K}$-polynomial function has a finite number of critical values.*

Proof. Let $C(f)$ be the set of critical points of the polynomial function f. The set $C(f)$ is a $\mathbb{K}$-algebraic subset of $\mathbb{K}^{n+1}$.

Let Σ_1 be the set of singular points of $C(f)$. It is an algebraic subset of $\Sigma_0 := C(f)$ (see [16] Lemma 2.2 and [21]).

We can define by induction Σ_i as the singular locus of Σ_{i-1}. It is a proper $\mathbb{K}$-algebraic subset of Σ_{i-1}:

$$\Sigma_i \subset \Sigma_{i-1}$$

and $\Sigma_{i-1} \setminus \Sigma_i$ is a manifold.

Hilbert finiteness theorem implies that the decreasing sequence of $\mathbb{K}$-algebraic sets $\Sigma_0 := C(f)$, Σ_1, …, Σ_i, … is finite. Therefore, we have a finite partition:

$$C(f) = (\Sigma_0 \setminus \Sigma_1) \coprod \ldots \coprod (\Sigma_{i-1} \setminus \Sigma_i) \coprod \ldots \coprod \Sigma_k,$$

which shows that the $\mathbb{K}$-algebraic subset $C(f)$ of $\mathbb{K}^{n+1}$ is the disjoint union of differences of $\mathbb{K}$-algebraic sets and each difference is a smooth manifold. Now, we have (see [16] Appendix A Corollary A.4):

Lemma 1.2 *The difference $E \setminus F$ of $\mathbb{K}$-algebraic subsets of $\mathbb{K}^{n+1}$, which is a smooth manifold, is the finite union of connected smooth manifolds.*

Let us assume the Lemma.

By the Lemma 1.2 the $\mathbb{K}$-algebraic subset $C(f)$ of $\mathbb{K}^{n+1}$ is a finite union of connected manifolds:

$$C(f) = V_1 \coprod \ldots\ldots \coprod V_\ell.$$

The restriction of f to any of these manifolds V_i, $1 \leq i \leq \ell$ is critical. Since V_i is connected, the restriction of f to V_i is constant. Therefore, the critical values of f are in finite number.

It remains to prove the Lemma.

We have to prove that the difference $E \setminus F$ of $\mathbb{K}$-algebraic subsets of $\mathbb{K}^{n+1}$, which is a smooth manifold, is the finite union of connected manifolds.

To prove this lemma, it is enough to consider the case $\mathbb{K} = \mathbb{R}$. Let $f_1, \ldots, f_r$ be real polynomials which define F on E:

$$F = E \cap \{f_1 = \ldots = f_r = 0\}.$$

Since the base field is the field of real numbers, the subset F is also defined by the polynomial $\sum_1^r f_i^2$:

$$F = E \cap \{\sum_1^r f_i^2 = 0\}.$$

Considering the map $\varphi : E \setminus F \to \mathbb{R}$ defined by:

$$p \mapsto \frac{1}{f_1^2(p) + \ldots + f_r^2(p)},$$

The projection onto E shows that the graph G of φ is diffeomorphic to $E \setminus F$.

The set G is also a real algebraic subset of $\mathbb{R}^{n+1} \times \mathbb{R}$ defined by the intersection:

$$(E \times \mathbb{R}) \cap \{T(\sum_1^r f_i^2) = 1\},$$

where T is the coordinate of $\mathbb{R}^{n+1} \times \mathbb{R}$ given by $\{0\} \times \mathbb{R}$.

Now the space G is a closed manifold embedded in $\mathbb{R}^{n+1} \times \mathbb{R}$. For almost all points p of $\mathbb{R}^{n+1} \times \mathbb{R}$, the square of the distance function to the point p is a non-degenerate function in the sense of Morse (see [17] Theorem 6.6). The critical points of such a function are isolated. They form a real algebraic subset of $\mathbb{R}^{n+1} \times \mathbb{R}$ whose points are isolated, so it must be a finite set (see [16] Appendix A, Corollary A.2). Therefore, by applying Morse theory [17], the space G has the homotopy type of a finite CW-complex. In particular, the number of connected components of G must be finite.

This ends the proof of the Lemma and therefore the proof of our Proposition.

1.2 Restriction to non-singular differences

Now, consider the restriction of a $\mathbb{K}$-polynomial function to a difference of $\mathbb{K}$-algebraic sets $E \setminus F$, which is a smooth manifold.

A result similar to Proposition 1.1 is also true. In fact the proof is analogous to the proof that we have just done:

Proposition 1.3 *The restriction of a $\mathbb{K}$-polynomial function to a difference of $\mathbb{K}$-algebraic sets $E \setminus F$ which is non-singular has a finite number of critical values.*

Proof: Let φ the restriction of the polynomial function $f : \mathbb{K}^N \to \mathbb{K}$ to E. Let us define the critical locus $C(\varphi)$ of φ as the union of the singular locus of E and the subset of the non-singular subset of E where φ is critical. The set $C(\varphi)$ is a $\mathbb{K}$-algebraic subset of E.

We just notice that the critical locus of the restriction of φ to $E \setminus F$ is a difference of $\mathbb{K}$-algebraic sets $C(\varphi) \setminus F$. Then, one proceeds as in the proof of Proposition 1.1.

One has:

$$C(\varphi) \setminus F = (\Sigma_0 \setminus (F \cup \Sigma_1)) \coprod \dots \coprod (\Sigma_{i-1} \setminus (F \cup \Sigma_i)) \coprod \dots \coprod (\Sigma_k \setminus F),$$

where $\Sigma_0 = C(\varphi)$, $\Sigma_i = \Sigma_{i-1}$ and $\Sigma_{k+1} = \emptyset$.

Lemma 1.2 tells that $C(\varphi) \setminus F$ is the disjoint union of a finite number of connected manifolds. The restriction of φ to each of these connected components is critical, so it must be constant. This implies Proposition 1.3.

1.3 Comments

The preceding result can be adapted to analytic functions. For instance, let $f : (\mathbb{K}^N, 0) \to (\mathbb{K}, 0)$ be the germ of a $\mathbb{K}$-analytic function. Then one can prove that there is $\varepsilon > 0$ such that there exists $\eta > 0$ for which the space $f^{-1}(t) \cap \overset{\circ}{\mathbb{B}}_\varepsilon(0)$ for t, $\eta > |t| > 0$, is a $\mathbb{K}$-analytic manifold.

We usually write that, let ε and η such that $1 \gg \varepsilon \gg \eta > 0$, then the space $f^{-1}(t) \cap \overset{\circ}{\mathbb{B}}_\varepsilon(0)$ is an analytic manifold for $\eta > |t| > 0$.

2 The fibration theorem (isolated singularity case)

2.1 The local link

Let us assume that the complex polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ has an isolated critical point at p_0. In this case we shall prove that we can associate a locally trivial smooth fibration to the critical point.

The difference $f^{-1}(f(p_0)) \setminus C(f)$ is a manifold. As a consequence of Proposition 1.3, the restriction of the square of the distance to p_0 to $f^{-1}(f(p_0)) \setminus C(f)$ has no critical value in the open interval $(0, \varepsilon_0)$ when ε_0 is a sufficiently small > 0 number. Therefore, for any ε, $\varepsilon_0 > \varepsilon > 0$, the intersection $\mathbb{S}_\varepsilon(p_0) \cap f^{-1}(f(p_0))$ of $f^{-1}(f(p_0))$ with the sphere $\mathbb{S}_\varepsilon(p_0)$ of $\mathbb{C}^{n+1}$ centered at p_0 with radius ε is a manifold $K_\varepsilon(p_0)$.

In fact, the square of the distance to the point p_0 defines a smooth map of $f^{-1}(f(p_0)) \setminus C(f)$ into the real line $\mathbb{R}$. The map induces a trivial fibration onto the open interval $(0, \varepsilon_0)$. The fibers of this fibration are the manifolds $K_\varepsilon(p_0)$ with ε, $\varepsilon_0 > \varepsilon > 0$. They are diffeomorphic between each other, as well as the pairs $(\mathbb{S}_\varepsilon(p_0), K_\varepsilon(p_0))$. This is why (by abuse of language) we call $K_\varepsilon(p_0)$ the *link* of the point p_0 on the hypersurface $f^{-1}f(p_0)$.

2.2 A fibration theorem

Let us choose ε, $\varepsilon_0 > \varepsilon > 0$. The Proposition 1.1 implies that there is η_ε, such that, for any η, $\eta_\varepsilon > \eta > 0$, the hypersurfaces $f^{-1}(f(p_0)+t)$ are non singular for any t, $\eta \geq |t| > 0$. Furthermore, since, by definition of $\varepsilon_0 > 0$, the sphere $\mathbb{S}_\varepsilon(p_0)$ intersects $f^{-1}(f(p_0))$ transversally in $\mathbb{C}^{n+1}$, by continuity, the hypersurfaces $f^{-1}(f(p_0)+t)$ are transverse in $\mathbb{C}^{n+1}$ to the sphere $\mathbb{S}_\varepsilon(p_0)$ for any t, $\eta \geq |t| > 0$.

We have:

Proposition 2.1 *Let us choose ε and η as above. The map:*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(p_0) \cap f^{-1}(\mathbb{S}^1_\eta(f(p_0))) \to \mathbb{S}^1_\eta(f(p_0))$$

induced by the polynomial function f, is a locally trivial smooth fibration over the circle $\mathbb{S}^1_\eta(f(p_0))$.

Proof. Since, for $z \in \mathbb{S}^1_\eta(f(p_0))$, the fibers $f^{-1}(z)$ are transverse in $\mathbb{C}^{n+1}$ to the sphere $\mathbb{S}_\varepsilon(p_0)$ the restriction of $\varphi_{\varepsilon,\eta}$ to the boundary of $\mathbb{B}_\varepsilon(p_0) \cap f^{-1}(\mathbb{S}^1_\eta(f(p_0)))$ is submersive. It is also obviously surjective. Since the fibers $f^{-1}(z)$ for $z \in \mathbb{S}^1_\eta(f(p_0)$ are non singular, the restriction of $\varphi_{\varepsilon,\eta}$ to the interior $\mathring{\mathbb{B}}_\varepsilon \cap f^{-1}(\mathbb{S}^1_\eta(f(p_0))$ is also submersive and surjective. We can conclude using Ehresmann Lemma.

2.3 Ehresmann Lemma

We have a lemma by C. Ehresmann (see e.g. [3] (20.8) problème 4, or see also [1] Theorem 8.12 p. 84 when the boundary is empty):

Lemma 2.2 (Ehresmann's Lemma) *Let $(M, \partial M)$ be a smooth manifold M with boundary ∂M. Let $\varphi : M \to N$ be a proper smooth map onto a connected manifold N. Assume:*

1. *The map φ is proper;*
2. *The restriction of φ to ∂M is submersive and surjective onto N;*
3. *The restriction of φ to M is submersive and surjective onto N.*

Then, the map φ is a locally trivial smooth fibration.

Using this lemma, we obtain immediately a proof of Proposition 2.1.

Exercise: Prove that, if $1 \gg \varepsilon \gg \eta > 0$, the different fibrations $\varphi_{\varepsilon,\eta}$ are isomorphic.

The fibration defined in Proposition 2.1 is called the *Milnor fibration* of the polynomial function f at p_0.

3 Stratifications

3.1 Algebraic partition

Let E be a complex algebraic subset of $\mathbb{C}^{n+1}$. Let $S_1, \dots, S_k$ be a finite partition of E:

$$E = S_1 \coprod \dots \coprod S_k.$$

We say that it is a *complex algebraic partition* if the closures $\overline{S}_i$ of S_i in $\mathbb{C}^{n+1}$ and differences $\overline{S}_i \setminus S_i$ are complex algebraic sets, for $1 \leq i \leq k$.

Similarly let E be a real semi-algebraic subset of $\mathbb{R}^N$. Let $S_1, \dots, S_k$ a finite partition of E:

$$E = S_1 \coprod \dots \coprod S_k.$$

We say that this partition is a *real algebraic partition* of E if the closures $\overline{S}_i$ of S_i in $\mathbb{R}^N$ and the differences $\overline{S}_i \setminus S_i$ are semi-algebraic sets, for all i, $1 \leq i \leq k$.

Any set S_i in a partition $\mathcal{S}$ of E, $1 \leq i \leq k$, is called a *stratum* of the partition.

3.2 Algebraic stratification

Let E be a complex algebraic subset of $\mathbb{C}^{n+1}$. Let $S_1, \dots, S_k$ be an algebraic partition of E.

We say that $(S_i)_{1 \leq i \leq k}$ is a *complex stratification*, if it is a complex algebraic partition and each S_i is a connected complex manifold and the partition satisfies the *frontier condition*, i.e. $S_i \cap \overline{S}_j \neq \emptyset$ implies $S_i \subset \overline{S}_j$.

Analogously let E be a real semi-algebraic subset of $\mathbb{R}^N$ and let $S_1, \dots, S_k$ be an algebraic partition of E.

We say that the partition $(S_i)_{1 \leq i \leq k}$ is a *real stratification* of E, if it is a real algebraic partition and each S_i is a connected smooth manifold and the partition satisfies the *frontier condition*, i.e. $S_i \cap \overline{S}_j \neq \emptyset$ implies $S_i \subset \overline{S}_j$.

Whenever we shall consider the stratification of a real algebraic set, we shall consider a stratification by semi-algebraic sets, considering a real algebraic set as a particular semi-algebraic set.

We say that a map:

$$\varphi : E \to F$$

of a stratified set $(E, \mathcal{S})$ into a stratified set $(F, \mathcal{T})$ is a *stratified map* if for any stratum $S_i \in \mathcal{S}$, there is a stratum $T_j \in \mathcal{T}$ such that φ induces a submersive and surjective map of S_i onto T_j.

3.3 An example

Let E be a $\mathbb{K}$-algebraic set. Let $\Sigma(E)$ be the subset of singular points of E. We can define a decreasing sequence of algebraic sets defined by induction:

1. $\Sigma_0 := E$

2. Let $i \geq 0$, $\Sigma_{i+1} = \Sigma(\Sigma_i)$

By Hilbert finiteness Theorem this sequence is stationary, so there is ℓ such that $\Sigma_\ell \neq \emptyset$ and $\Sigma_{\ell+1} = \Sigma(\Sigma_\ell) = \emptyset$. Therefore, we have:

$$E = (\Sigma_0 \setminus \Sigma_1) \coprod \dots \coprod (\Sigma_i \setminus \Sigma_{i+1}) \coprod \dots \coprod \Sigma_\ell.$$

In this way, we have defined a partition of E defined by the strata $\Sigma_i \setminus \Sigma_{i+1}$, for $1 \leq i \leq \ell + 1$.

The strata are smooth manifolds. However, the partition might not satisfy the frontier condition.

An algebraic partition $\mathcal{T}$ of a $\mathbb{K}$-algebraic set is said to be *finer* than the algebraic partition $\mathcal{S}$ if the closures of the strata of $\mathcal{S}$ are union of strata of $\mathcal{T}$. We also say that the partition $\mathcal{T}$ is a *refinement* of $\mathcal{S}$.

In [22] (Theorem 18.11) H. Whitney proves that:

Proposition 3.1 *Any stratification of a $\mathbb{K}$-algebraic set has a refinement which is a stratification with connected strata.*

With these definitions we can define regular stratifications (see [22]), i.e. Whitney stratifications.

4 Whitney stratifications

4.1 The condition (a) of Whitney

Let $\mathcal{S} = (S_i)_{1 \leq i \leq k}$ be a stratification of a $\mathbb{K}$-algebraic subset E of $\mathbb{K}^N$. Let (S_i, S_j) be a pair of strata such that $S_i \subset \overline{S}_j$. Let $p \in S_i$. We say that the pair (S_i, S_j) satisfies the condition (a) of Whitney along S_i at the point p, if, for any sequence (p_n) of points of S_j which converges to p such the sequence of tangent spaces $(T_{p_n} S_j)$ has a limit T, then the tangent space $T_p S_i$ is contained in T.

We say that, the pair (S_i, S_j) satisfies the *condition (a) of Whitney* along S_i, if it satisfies the condition (a) of Whitney along S_i at any point p of S_i.

Example: Consider the complex algebraic set E defined by $X^2 - Y^2 Z = 0$, which is known as "Whitney umbrella".

The singular set of E is given by $X = Y = 0$. This is a line Σ contained in E. A stratification of E is given by $E \setminus \Sigma$ and Σ. The pair $(\Sigma, E \setminus \Sigma)$ does not satisfy the condition (a) of Whitney along Σ at the origin 0 of $\mathbb{C}^3$.

One may consider a sequence (p_n) of points of $E \setminus \Sigma$ given by $p_n = (0, y_n, 0)$, where $\lim_{n \to \infty} y_n = 0$. Then, $p = (0,0,0)$. We have $T_{p_n}(E \setminus \Sigma) = (X, Y) -$ plane and $T_p(\Sigma) = Z -$ axis. The limit at the point $(0,0,0)$ of the sequence $(T_{p_n}(E \setminus \Sigma))_{n \in \mathbb{N}}$ of tangent spaces is the $(X, Y) -$ plane and it does not contain the $Z -$ axis.

4.2 The condition (b) of Whitney

Let (S_i, S_j) be a pair of strata such that $S_i \subset \overline{S}_j$. Let $p \in S_i$. We say that the pair (S_i, S_j) satisfies the condition (b) of Whitney along S_i at the point p, if for any sequence p_n of points of S_j and any sequence (q_n) of points of S_i, which converges to p, such that the sequence of tangent spaces $T_{p_n}S_j$ has a limit T and the sequence of lines $q_n p_n$ has a limit ℓ, we have $\ell \subset T$.

We say that the pair (S_i, S_j) satisfies the *condition* (b) *of Whitney* along S_i if it satisfies the condition (b) of Whitney along S_i at any point p of S_i.

Examples: In the example given above the pair $(\Sigma, E \setminus \Sigma)$ does not satisfy the condition (b) of Whitney along Σ at the origin 0 of $\mathbb{C}^3$.

Consider the complex algebraic set F given by $X^2 - Y^2Z^2 - Z^3 = 0$ in $\mathbb{C}^3$. The singular locus Σ is the line $X = Z = 0$. One can prove that this stratification satisfies the condition (a) of Whitney, but it does not satisfies the condition (b) at the origin 0.

The sequences $p_n = (0, 1/n, -1/n^2)$ and $q_n = (0, 1/n, 0)$ give sequences of tangent planes $(T_{p_n}(F \setminus \Sigma))$ and of lines $(q_n p_n)$ whose limits T and ℓ are such that: $\ell \not\subset T$.

In fact, the condition (b) of Whitney implies the condition (a) of Whitney (see [14] Proposition 2.4):

Lemma 4.1 *Let E be a complex analytic subset of $\mathbb{C}^N$ (resp. a real semi-algebraic subset of $\mathbb{R}^N$). Let $\mathcal{S} = (S_1, \dots, S_k)$ be a stratification of E. Suppose that $S_i \subset \overline{S}_j$. Let $p \in S_i$. Suppose that the pair (S_i, S_j) satisfies the (b)-condition of Whitney at p, then it satisfies the (a)-condition of Whitney at p.*

Proof. The following proof was given to me orally by D. Cheniot.

Assume that the pair (S_i, S_j) satisfies the condition (b) of Whitney at $p \in S_i$. Consider a sequence $(p_n)_{n \in \mathbb{N}}$ a sequence of point of S_j which converges to p and such that the sequence of tangent spaces $T_{p_n}(S_j)$ converge to T. We have to show that: $T_p(S_i) \subset T$. Suppose otherwise. Then, there is a line D in $\mathbb{K}^N$ through the origin, such that $D \subset T_p(S_i)$, but $D \not\subset T$.

By definition of the tangent space $T_p(S_i)$, there is a sequence of points $(q_n)_{n \in \mathbb{N}}$ of points of S_i which converges to p and such that the lines pq_n converge to D. Since the sequence (p_n) converges to p, endowing the projective space of line directions with a metric, for any $k \in \mathbb{N}$, we can find n_k such that the distance between the line directions pq_k and $q_k p_{n_k}$ is bounded by $1/k$. Therefore, $q_k p_{n_k}$ converges to D, and this would contradict that (S_i, S_j) satisfies the condition (b) of Whitney at $p \in S_i$.

4.3 Whitney condition

Definition 4.2 *We say that a stratification $\mathcal{S}$ of a $\mathbb{K}$-algebraic set satisfies the Whitney condition (resp. the Whitney condition (a)) if, for any pair (S_i, S_j) of strata of $\mathcal{S}$ such that $S_i \subset \overline{S}_j$, the pair satisfies the Whitney condition (b) (resp. the Whitney condition (a)).*

In [22] (Theorem 19.2) H. Whitney proves:

Theorem 4.3 *Any $\mathbb{K}$-algebraic set has a Whitney stratification. Any stratification has a refinement which is a Whitney stratification.*

Note: One can define partitions and stratifications of $\mathbb{K}$-analytic subsets of open subsets of $\mathbb{K}^N$. Also one can define the Whitney conditions in this context. See the original paper of H. Whitney [22].

We can also extend the notion of stratification to any complex algebraic variety V in the following way. An algebraic partition of V is a finite partition $S_1, \ldots, , S_k$ such that the closures $\overline{S}_i$ of S_i in V and the differences $\overline{S}_i \setminus S_i$ are subvarieties of V, for $1 \leq i \leq k$.

An algebraic partition of complex algebraic variety is a stratification if each S_i is a manifold and the partition satisfies the frontier condition, i.e. $S_i \cap \overline{S}_j \neq \emptyset$ implies $S_i \subset \overline{S}_j$.

4.4 Characterization of Whitney stratifications by Polar varieties

In the case the base field is the field of complex numbers $\mathbb{C}$, there are many ways to characterize Whitney stratifications. To my knowledge there are no such characterization over the real numbers.

One way is to consider polar varieties (see [19]).

Let E be an equidimensional complex algebraic subset of $\mathbb{C}^N$. Let p_0 be a point of E. Consider the restriction of a linear projection to E:

$$\pi_k : E \to \mathbb{C}^{k+1}.$$

The restriction of π_k to the non-singular part $E \setminus \Sigma(E)$ of E has a critical set Γ_k. We have (see [12] (2.2.2)):

Proposition 4.4 *For all projections of a Zariski dense open set Ω_k in the space of linear projections of $\mathbb{C}^N$ onto $\mathbb{C}^{k+1}$, the closure $P_k = \overline{\Gamma}_k$ of Γ_k is either empty or a complex algebraic set of dimension k and the multiplicity of P_k is the same for any $\pi_k \in \Omega_k$.*

Definition 4.5 *For $1 \leq k \leq \dim E$, the algebraic sets defined by the projections $\pi_k \in \Omega_k$ are called the polar varieties of E at the point p_0 and the multiplicity of P_k at p_0 is called the k-th polar multiplicity of E at p_0.*

In [19] B. Teissier proved the following:

Theorem 4.6 *Let E be an equisingular complex algebraic set. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a stratification of E. Assume that for any pair of strata (S_i, S_j) such that $S_i \subset \overline{S}_j$ the k-th polar multiplicity of the closure $\overline{S}_j$ is the same at all the points of S_i, when $1 \leq k \leq \dim S_j$, then $\mathcal{S}$ is a Whitney stratification of E.*

There is also a combinatorial way to state if a stratification is a Whitney stratification. We shall introduce this way later because we need to understand first how much Milnor fibration Theorem can be extended.

4.5 First isotopy theorem of Thom-Mather

The main properties of Whitney stratifications are Thom-Mather isotopy Theorems ([20] and [14] Porposition 11.1).

The first isotopy theorem can be understood as an extension of Ehresmann Lemma that we have used to show Milnor fibration Theorem for hypersurfaces with an isolated singular point.

In many cases we shall consider real algebraic sets and more generally semi-algebraic sets. In the theorem below V might be a semi-algebraic set or a complex algebraic set.

Theorem 4.7 (First isotopy theorem of Thom-Mather) *Let $\varphi : V \to T$ be a proper algebraic map. Assume that V is stratified by $\mathcal{S} = (S_1, \ldots, S_k)$ and that T is connected and non-singular. If the restrictions of φ to each strata S_i $(1 \leq i \leq k)$ is a smooth map which are submersive and surjective onto T, then the map φ is a locally trivial topological fibration.*

Example. Let E be the real algebraic subset of $\mathbb{R}^3$ defined by:

$$XY(X-Y)(X-TY) = 0.$$

Consider the projection onto the T-axis. Consider the "tube" given by $X^2 + Y^2 \leq 1$. The intersection:

$$V = E \cap \{X^2 + Y^2 \leq 1\} \cap \{0 < T < 1\}$$

is semi-algebraic. The projection φ onto the T-axis induces a proper map of V onto the open interval $0 < T < 1$. Let us stratify V with $S_1 = \{(0,0)\} \times \{0 < T < 1\}$, $S_2 = V \cap \{X^2 + Y^2 = 1\}$ and $S_3 = V \setminus (S_1 \cup S_2)$. The restrictions of φ to the connected components of each of S_i, $i = 1, 2, 3$, induce submersive and surjective maps onto the open interval $\{0 < T < 1\}$.

The first isotopy theorem of Thom-Mather tells us that φ is a locally trivial continuous fibration, but it cannot be a smooth fibration otherwise the cross ratio of the lines:

$$XY(X-Y)(X-tY) = 0$$

which varies continuously with t would be constant.

5 Thom Condition

5.1 Definition

When one considers a map f between stratified sets, there is an important condition on the stratification of the source called Thom condition or a_f condition (see [20] or [14] p. 65).

Let $\varphi : V \to W$ be an algebraic map. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a stratification of V.

We say that the stratification $\mathcal{S}$ of V satisfies the *Thom condition* or the *a_φ condition* if, for any pair (S_i, S_j) of strata, such that $S_i \subset \overline{S}_j$, for any point $p \in S_i$ and any sequence q_n of points of S_j converging to p for which the limit of the tangent spaces at q_n to the fibers $\varphi^{-1}(\varphi(q_n)) \cap S_j$ exists and is equal to T, we have $T \supset T_p(\varphi^{-1}(\varphi(p)) \cap S_i)$.

Example. Let $e : E \to \mathbb{C}^2$ be the blowing-up of the point 0 in the complex plane $\mathbb{C}^2$. Let stratify the map e by $E \setminus D$ and D, where D is the exceptional divisor of e and $\{0\}$ and $\mathbb{C}^2 \setminus \{0\}$. This stratified map does not satisfy the Thom condition, since the fibers of e outside 0 are points.

Note that, pulling back the map e by e itself, the new map that we obtain can be stratified to satisfy the Thom condition (see [18]).

5.2 Hironaka Theorem

A theorem of H. Hironaka will allow us to generalize Milnor fibration Theorem (see [6] Corollary 1 §5 p. 248):

Theorem 5.1 *Let $f : E \to C$ be an algebraic map from a complex algebraic set E into a non-singular complex curve. One can stratify the map f by Whitney stratifications such that the stratification of E satisfies Thom condition for f.*

In the case of hypersurfaces in $\mathbb{C}^{n+1}$, in [5] (Theorem 1.2.1 p. 322 - 324) we prove this theorem using Łojasiewicz inequality.

5.3 An example

In general when the target of the map has not complex dimension one, there are no reason to obtain stratifications with Thom conditions.

For instance, consider the polynomial map $F : \mathbb{C}^3 \to \mathbb{C}^2$ given by:

$$F(X, Y, Z) = (Y^2 - X^2 Z, X).$$

The critical locus is $Y = X^2 = 0$. Its image by F is $(0, 0)$.

One can check that at the origin one cannot find a Whitney stratification which satisfies Thom condition.

In fact, in [18] it is proved that after a base change the original map gives a new map which can be stratified with Thom condition. This result has been little used in the litterature.

5.4 Second isotopy theorem of Thom-Mather

Now we can formulate the second isotopy Theorem of Thom-Mather (see [14] Proposition 11,2).

Consider the following diagram:

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "f\circ g"'] & & F \arrow[dl, "f"] \\ & T & \end{tikzcd}$$

where g is a stratified map, where E is stratified by $\mathcal{S} = (S_1, \ldots, S_k)$ and F is stratified by $\mathcal{S}' = (S'_1, \ldots, S'_\ell)$, and T is non-singular. We say that g is a *Thom map over T* if:

1. The maps g and f are proper;
2. For each stratum S'_i of $\mathcal{S}'$ the restriction $f|S'_i$ is submersive;
3. For any stratum S_j of $\mathcal{S}$ there is a stratum S'_i of $\mathcal{S}'$ such that $g(S_j) \subset S'_i$ and the map induced by g from S_j into S'_i is submersive;
4. The stratification $\mathcal{S}$ satisfies the Thom condition relatively to g.

Then:

Theorem 5.2 (Second isotopy theorem of Thom-Mather) *If g is a Thom map over T, the map g is topologically locally trivial over T.*

6 The fibration theorem

6.1 Case of functions in $\mathbb{C}^{n+1}$

In §2 we have proved a fibration theorem when the point p_0 is an isolated critical point of the polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$.

In this section we shall drop the hypothesis that the point p_0 is an isolated critical point.

Since the critical point is no more isolated, the critical point locus of f intersects any sphere centered at the point p_o. Therefore, the hypersurface $f^{-1}(f(p_0))$ does no intersect transversally in $\mathbb{C}^{n+1}$ small spheres $\mathbb{S}_\varepsilon(p_0)$. However, by Proposition 1.1, the fibers $f^{-1}(p_0 + t)$ for t small enough and $\neq 0$ are non-singular hypersurfaces. Hironaka Theorem (see 5.1 above) tells us that we can stratify the polynomial f by Whitney stratifications with Thom condition.

Let us fix a Whitney stratification $\mathcal{S} = (S_1, \dots, S_k)$ of $\mathbb{C}^{n+1}$ adapted to $f^{-1}(f(p_0))$ and which satisfies Thom condition relatively to the polynomial function f.

We have seen that there is $\varepsilon_i > 0$ such that the restriction to a stratum S_i, which is the difference of complex algebraic set, of the square of the distance function to the point p_0 has no critical point in an open interval $(0, \varepsilon_i)$ (see Proposition 1.3 above). Therefore, there is $\varepsilon_0 = \inf_{1 \leq i \leq k}(\varepsilon_i) > 0$, such that for ε, $\varepsilon_0 > \varepsilon > 0$, the sphere $\mathbb{S}_\varepsilon(p_0)$ intersects all the strata of $\mathcal{S}$ transversally in $\mathbb{C}^{n+1}$. Since the stratification $\mathcal{S}$ satisfies Thom condition relatively to f, there is $\eta > 0$ small enough, such that the non-singular hypersurfaces $f^{-1}(f(p_0 + t))$ intersect the sphere $\mathbb{S}_\varepsilon(p_0)$ transversally in $\mathbb{C}^{n+1}$, when $\eta > |t| > 0$.

We can apply Ehresmann Lemma as in §2 to claim that:

Theorem 6.1 *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a complex polynomial function. Let p_0 be point of $\mathbb{C}^{n+1}$. There is $\varepsilon_0 > 0$, such that for any ε, $\varepsilon_0 > \varepsilon > 0$, there is η_0, such that, for any $\eta > 0$, $\eta_0 > \eta > 0$, the map:*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(p_0) \cap f^{-1}(\mathbf{S}^1_\eta(f(p_0))) \to \mathbf{S}^1_\eta(f(p_0))$$

induced by the polynomial function f is a locally trivial differentiable fibration.

6.2 Functions on an algebraic set

In fact, if we use the Theorem 5.1 we can obtain a more general statement, but we have only to consider topological fibrations instead of differentiable fibrations (see [9] Theorem 1.1):

Theorem 6.2 *Let E be a complex algebraic subset of $\mathbb{C}^N$ and p_0 be a point of E. Let $f : \mathbb{C}^N \to \mathbb{C}$ be a complex polynomial. There is $\varepsilon_0 > 0$, such that, for any ε, $\varepsilon_0 > \varepsilon > 0$, there is η_0, such that, for any $\eta > 0$, $\eta_0 > \eta > 0$, the map:*

$$\varphi_{\varepsilon,\eta} : \mathbb{B}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbf{S}^1_\eta(f(p_0))) \to \mathbf{S}^1_\eta(f(p_0))$$

induced by the polynomial function f is a locally trivial continuous fibration.

Since we can stratify the restriction $f_E : E \to \mathbb{C}$ of f to E to have Whitney and Thom conditions, for every strata S_i which has p_0 in its closure, we can find $\varepsilon_0 > 0$, such that, for any ε, $\varepsilon_0 > \varepsilon > 0$, the sphere $\mathbb{S}_\varepsilon(p_0)$ intersect the stratum S_i (i=1,..., k) transversally in $\mathbb{C}^N$, because each S_i is a difference of complex algebraic sets (see Proposition 1.3 of §1 above).

Let us fix such a $\varepsilon > 0$. Because of Thom condition there is $\eta_0 > 0$ such that the fiber:

$$f^{-1}(f(p_0 + t)) \cap S_i$$

intersects transversally the sphere $\mathbb{S}_\varepsilon(p_0)$ in $\mathbb{C}^N$. We can stratify the semi-algebraic set:

$$\mathbb{B}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbf{S}^1_\eta(f(p_0)))$$

with the strata:

$$S_i \cap \mathring{\mathbb{B}}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbf{S}^1_\eta(f(p_0)))$$

and $S_i \cap \mathbb{S}_\varepsilon(p_0) \cap E \cap f^{-1}(\mathbf{S}^1_\eta(f(p_0)))$, for $i = 1, \ldots, k$. By using the result of [2], we can prove that this stratification satisfies the frontier condition, then, it is easy to prove that it is a Whitney stratification.

The first isotopy Theorem of Thom-Mather 4.7 (instead of Ehresmann Lemma) implies our Theorem.

Note and Exercise: The Theorem 6.2 is true when one considers a complex analytic function on a complex analytic set. To prove it, one needs stratifications of complex analytic sets. In this case the number of strata might not be finite, but the partition is required to be locally finite. We leave the reader to develop by himself the correct theory of stratifications. The isotopy theorems are true for a general theory of stratified sets.

7 Relative Polar curve and general projections

7.1 Isolated singularity relatively to a stratification

Let E be a complex algebraic subset of $\mathbb{C}^N$ and p_0 be a point of E.

Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a Whitney stratification. The restriction $F = f\,|\,E$ of a complex polynomial f to the complex algebraic set E has an *isolated singularity at the point p_0 relatively to* *the stratification* $\mathcal{S}$ if the fiber $f^{-1}(f(p_0))$ intersects the strata S_i transversally in $\mathbb{C}^N$ at any point outside p_0 (see [10]).

One can observe that if the restriction F of a complex polynomial function f to E has an isolated singular point relatively to the Whitney stratification $\mathcal{S}$, the stratification automatically satisfies Thom condition relatively to F.

7.2 Relative polar curve

Now, let $l : \mathbb{C}^N \to \mathbb{C}$ be a linear form. Consider the restriction ℓ of l to E. The functions F and ℓ define a map:

$$\Phi = (\ell, F) : E \to \mathbb{C}^2.$$

Observe that, if the linear form l is sufficiently general, the map Φ satisfies Thom condition relatively to the Whitney stratification of E.

In fact, if the linear form l is sufficiently general, the fibers of Φ have isolated singularities relatively to the stratification $\mathcal{S}$ in the sense that, for any point $x \in E$, the fiber $\tilde{\Phi}^{-1}(\Phi(x))$ intersects the strata S_i transversally in $\mathbb{C}^N$ except at a finite number of points, where $\tilde{\Phi}$ is the polynomial map:

$$(l, f) : \mathbb{C}^N \to \mathbb{C}^2.$$

Precisely, consider the restriction of Φ to each of the strata S_i, $1 \leq i \leq k$. If the linear form l is sufficiently general, the critical locus Γ_i is either the empty set $\emptyset$ or a curve. The set:

$$\Gamma(l) = \cup_1^k \overline{\Gamma}_i,$$

which is either empty or an algebraic curve, is called the *relative polar curve* of the linear form l relatively to the stratification $\mathcal{S}$. We have:.

Lemma 7.1 *The restriction of Φ to the germ at a point $p \in E$ of the relative polar curve $\Gamma(l)$ is quasi-finite.*

This lemma is an immediate consequence of the fact that the fiber $\Phi^{-1}(\Phi(p))$ has an isolated singularity relatively to the stratification $\mathcal{S}$. Then, the fiber of the germ of Φ restricted to $\Gamma(l)$ is either empty or equal to $\{p\}$.

This observation implies:

Corollary 7.2 *The germ of Φ at ay point $p \in E$ restricted to the germ $\Gamma(l)$ at p is a germ of finite complex analytic map.*

This corollary is an immediate consequence of the lemma by using the geometric version of Weierstrass Preparation theorem given by C. Houzel in [7] which tells that a complex analytic map germ with finite fibers is finite.

This means that there are neighborhoods U and V of p and $\Phi(p)$ such that Φ induces a finite analytic map, i.e. a proper complex analytic map with finite fibers, of $U \cap \Gamma(l)$ into V.

7.3 Generalized discriminant

We keep the notations of the preceding section. Then, we have:

Proposition 7.3 *There are open neighborhoods $U(p)$ and $V(p)$ of p and $\Phi(p)$ such that the map Φ induces a map $\Phi_p : U(p) \to V(p)$ such that the restriction of Φ_p to $\Gamma(l) \cap U(p)$ is a complex analytic finite map, and Φ_p induces a locally trivial topological fibration of $U(p) \setminus \Phi_p^{-1}(\Phi_p(\Gamma(l) \cap U(p)))$ onto $V(p) \setminus \Phi_p(\Gamma(l) \cap U(p))$.*

As we have seen before, Corollary 7.2 implies that there are neighborhoods U and V of p and $\Phi(p)$ such that Φ induces a finite map $\Phi_p : U \to V$, therefore, the image $\Delta(l) = \Phi_p(\Gamma(l) \cap U)$ is a complex analytic curve of V.

By definition inside U the restriction of Φ to any strata S_i has no critical point at any point of $\Phi^{-1}(y)$ with $y \in V \setminus \Delta(l)$. Let $\mathbb{S}_\varepsilon(p)$ be the sphere of $\mathbb{C}^N$ centered at the point p with radius ε such that the ball $\mathbb{B}_\varepsilon(p)$ that it bounds is contained in U and any sphere $\mathbb{S}_{\varepsilon'}(p)$ of radius $\varepsilon \geq \varepsilon' > 0$ is transverse in $\mathbb{C}^N$ to $\Phi^{-1}(\Phi(p)) \cap (S_i \cap U)$, for $1 \leq i \leq k$.

Since the fiber $\Phi^{-1}(\Phi(p))$ has an isolated singularity at p relatively to the stratification $\mathcal{S}$, there is $\eta > 0$ such that, for all $y \in \overset{\circ}{\mathbb{B}}_\eta(\Phi(p))$, the fiber $\Phi^{-1}(y)$ is transverse in $\mathbb{C}^N$ to $\mathbb{S}_\varepsilon(p)$.

Define $U(p) = \overset{\circ}{\mathbb{B}}_\varepsilon(p) \cap E \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)))$ and $V(p) = \overset{\circ}{\mathbb{B}}_\eta(\Phi(p))$ and the map Φ induces a map:

$$\Phi_p : U(p) \to V(p).$$

The sets $S_i \cap \overset{\circ}{\mathbb{B}}_\varepsilon(p) \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)) \setminus \Delta(l))$ and $S_i \cap \mathbb{S}_\varepsilon(p) \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)) \setminus \Delta(l))$ define a Whitney stratification of $\mathbb{B}_\varepsilon(p) \cap E \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)))$ (use [2] to prove the frontier property, then it is easy to prove that one has Whitney condition) and Φ induces a map:

$$\varphi : \mathbb{B}_\varepsilon(p) \cap E \cap \Phi^{-1}(V(p) \setminus \Delta(l)) \to V(p) \setminus \Delta(l).$$

Since $\mathbb{B}_\varepsilon(p)$ is compact, the map φ is proper and we have defined $\Gamma(l)$ and $\Delta(l)$ such that it has maximal rank on the Whitney strata of:

$$\mathbb{B}_\varepsilon(p) \cap E \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)) \setminus \Delta(l)).$$

Thom-Mather first isotopy Theorem implies that φ is a locally trivial topological fibration. Therefore, this fibration induces a locally trivial topological fibration of $U(p) \cap \Phi^{-1}(V(p) \setminus \Delta(l))$ over $V(p) \setminus \Delta(l)$.

Then, one may stratify the map $\Phi_p : U(p) \to V(p)$ by considering on $U(p)$ the strata $S_i \cap U(p) \setminus \Gamma_i$, $(\Gamma_i \setminus \{p\}) \cap U(p)$ and $\{p\}$ and on $V(p)$ the strata $V(p) \setminus \Delta(l)$, $\Phi((\Gamma_i \setminus \{p\}) \cap U(p))$ and $\{\Phi(p)\}$. These stratifications are Whitney stratifications since $\mathcal{S}$ is a Whitney stratification and $V(p)$ is non-singular.

Locally, the set $\Delta(l) \cap V(p)$ plays the role of a discriminant of the complex analytic map Φ_p. We shall call it the *topological discriminant* of Φ.

Note: One can define by induction a map $\Phi_{r+1} = (\ell_1, \ldots, \ell_r, F) : E \to \mathbb{C}^{r+1}$ for $r \leq \dim E$ where $\ell_1, \ldots, \ell_r$ are restrictions to E of general linear forms of $\mathbb{C}^N$. For any point p of E, the map Φ_{r+1}

induces a map of an open neighborhood $U(p)$ of p in E into an open neighborhood $V(p)$ of $\Phi(p)$ in $\mathbb{C}^{r+1}$.

The union of the images by Φ_{r+1} of the closures in E critical loci of Φ_{r+1} restricted to the strata S_i would locally define a *topological* discriminant $\Delta \cap V(p)$ of Φ_{r+1}.

Let us suppose that the function F is itself induced by a general linear for l_0. In this way we can define a *general projection* of E into $\mathbb{C}^{r+1}$.

8 Vanishing Euler characteristics

8.1 Local Vanishing fibers

Let E be a complex algebraic subset of $\mathbb{C}^N$ and p_0 be a point of E.

We have defined a general projection $\pi_r : E \to \mathbb{C}^r$ for any r, $1 \leq r \leq \dim E$. At the point p_0 we have defined a map germ:

$$\pi_r : (E, p_0) \to (\mathbb{C}^r, \pi_r(p_0))$$

and a germ of topological discriminant $(\Delta(\pi_r), \pi_r(p_0))$. The map germ π_r induces a map:

$$p_r : U(p_0) \to V(p_0)$$

and a topological discriminant Δ_r which is a closed subset of $V(p_0)$. The map p_r induces a locally trivial topological fibration of $U(p_0) \setminus p_r^{-1}(V(p_0) \cap \Delta_r)$ onto $V(p_0) \setminus \Delta_r$. The fiber of this locally trivial fibration is called a *vanishing fiber* of dimension $\dim E - r$ of E at p_0.

The Euler characteristic $\chi_{\dim E - r}(p_0)$ of a vanishing fiber of dimension $\dim E - r$ at the point p_0 is called the *vanishing Euler characteristic* of E at p_0 of dimension $\dim E - r$.

At any point p of a complex algebraic set E we define a $(\dim E)$-uple:

$$\chi_*(E, p) = (\chi_0(p), \ldots, \chi_{\dim E - 1}(p)).$$

We may call $\chi_*(E, p)$ the *vanishing characteristics* of E at p.

8.2 Combinatorial characterization of Whitney conditions

We can formulate the following characterization of Whitney conditions (see [13] Théorème (5.3.1)):

Theorem 8.1 *Let E be a complex algebraic subset of $\mathbb{C}^N$. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a stratification of E. Suppose that, for any pair (S_i, S_j) such that $S_i \subset \overline{S}_j$, the vanishing characteristics $\chi_*(\overline{S}_j, p)$ is constant for $p \in S_i$, then the stratification $\mathcal{S}$ is a Whitney stratification.*

This theorem suggests that there is a relation between Vanishing characteristics and Polar varieties.

This relation exists when one considers a Whitney stratification $\mathcal{S} = (S_1, \ldots, S_k)$ of E. It is convenient to write $\chi(E, S_i)$ (or $\chi(\overline{S}_j, S_i)$, whenever $S_i \subset \overline{S}_j$) for $\chi(E, p)$ (or for $\chi(\overline{S}_j, p)$) with $p \in S_i$, since one can prove that, using Thom-Mather first isotopy theorem, the topology of the vanishing fibers of E (or $\overline{S}_j$) along S_i does not vary.

In particular we have (see [13] Théorème (4.1.1)):

Theorem 8.2 *Let E be an equidimensional complex algebraic subset of $\mathbb{C}^N$. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a Whitney stratification of E. Let us denote $d_i = \dim S_i$. Let $x \in S_i$. We have:*

$$\begin{aligned} &\chi_{\dim E - d_i - 1}(E, S_i) - \chi_{\dim E - d_i - 2}(E, S_i) \\ &= \sum_{\mathrm{j} \neq i} (-1)^{d_j - d_i - 1} m(P_{\dim E - d_j + d_i + 1}(\overline{S}_j), x))(1 - \chi_{\dim E - d_j - 1}(X, X_j)) \end{aligned}$$

where $m(P, x)$ is the multiplicity of P at x.

The study of Whitney stratification leads, on the algebraic geometry side, to the study of Polar Varieties and, on the topological side, to Vanishing Fibers or Vanishing Euler characteristics.

9 Bouquet of spheres

9.1 Statement of the theorem

In some situations, results can be formulated concerning the Vanishing Fibers associated to germs of complex algebraic sets.

The most surprising result is (see [11]):

Theorem 9.1 *The vanishing Fibers of a germ of complex complete intersection have the homotopy type of a bouquet of real spheres of dimension equal to the complex dimension of each Vanishing Fiber.*

Just notice that this is a theorem similar to the result of J. Milnor in [16] (Theorem 6.5) or to the one of H. Hamm for the case of complete intersections ([4]).

9.2 Hypersurfaces

In this section we shall first assume that E is a complex hypersurface define by the polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$. In this section f might have non-isolated singularities.

We have seen that we can stratify the map f with Whitney stratifications satisfying Thom condition (see above Theorem 5.1 by Hironaka).

Let l be a general linear form of $\mathbb{C}^{n+1}$. It defines a map:

$$\Phi = (l, f) : \mathbb{C}^{n+1} \to \mathbb{C}.$$

Let p be a point of E. Consider the germ of Φ at p. The critical locus of the germ of Φ at p depends linearly on l. A classical theorem of Bertini says that the singular points of this linear system lie in the fixed points of the linear system which are precisely the critical space of f at p.

Therefore in a sufficiently small neighborhood U of p in $\mathbb{C}^{n+1}$ the critical space of Φ is non singular outside the hypersurface E. The closure in U of this critical space is empty or a curve Γ that we have called the relative polar curve of f relatively to the linear form l in the case f has an isolated singularity at p..

As we have done in §7, we can prove that there are $1 \gg \varepsilon \gg \eta > 0$ such that Φ induces:

$$\Phi_{\varepsilon,\eta}(p) : \mathbb{B}_\varepsilon(p) \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p))) \to \overset{\circ}{\mathbb{B}}_\eta(\Phi(p))$$

which is a stratified map with Thom condition. Let Δ be the union of the image of the complex curve $\Gamma \cap \mathbb{B}_\varepsilon(p) \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)))$ by $\Phi_{\varepsilon,\eta}(p)$ and the trace in $\overset{\circ}{\mathbb{B}}_\eta(\Phi(p))$ of the line $\mathbb{C} \times \{f(p)\}$:

$$\Delta := \Phi_{\varepsilon,\eta}(p)(\Gamma \cap \mathbb{B}_\varepsilon(p) \cap \Phi^{-1}(\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)))) \cup (\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)) \cap \mathbb{C} \times \{f(p)\}).$$

As we have done in §7, we can prove that the map $\Phi_{\varepsilon,\eta}(p)$ induces a locally trivial smooth fibration on $\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)) \setminus \Delta$.

Notice that the germ of the critical locus of f at p which might be of dimension ≥ 1 has its image by the germ of Φ at p contained in $\mathbb{C} \times \{f(p)\}$.

Since the linear form l is general, if it is not empty, the space $\Gamma \setminus \{p\}$ is non-singular, which means that the fiber $\Phi_{\varepsilon,\eta}(p)^{-1}(\Phi(p'))$ has an ordinary quadratic point at $p' \in \Gamma \setminus \{p\}$. This shows that the fibers of $\Phi_{\varepsilon,\eta}(p)$ over $\overset{\circ}{\mathbb{B}}_\eta(\Phi(p)) \setminus \mathbb{C} \times \{0\}$ are transverse to $\mathbb{S}_\varepsilon(p)$ in $\mathbb{C}^{n+1}$, because the fibers over $\Delta \setminus (\mathbb{C} \times \{f(p)\})$ have an isolated singularity at the points of $\Gamma \setminus \{p\}$.

For t sufficiently small, the fiber of $\Phi_{\varepsilon,\eta}(p)$ above $(l(p)+t, f(p))$, is a vanishing fiber of dimension $\dim E - 1$ of E at p. Let us call (u, v) the coordinates of $\mathbb{C}^2$ such that;

$$u = l \text{ and } v = f.$$

Consider the line $u = l(p) + t$. One can prove that the space of complex dimension n:

$$\Phi_{\varepsilon,\eta}(p)^{-1}(\{u = l(p) + t\})$$

is contractible and the space:

$$\Phi_{\varepsilon,\eta}(p)^{-1}(\mathbb{D})$$

where $\mathbb{D}$ is a small disc of radius r of the line $\{u = l(p) + t\}$ centered at $(l(p) + t, f(p))$ retracts by deformation on the vanishing fiber $\Phi_{\varepsilon,\eta}(p)^{-1}((l(p) + t, f(p)))$.

The restriction of $|f|$ to the space $\Phi_{\varepsilon,\eta}(p)^{-1}(\{u = l(p)+t\})$ defines a real function, Let us start with the value r. As the values of $|f|$ meet the values of the intersection points of the line $(\{u = l(p)+t\})$ with Δ, the restriction of $|f|$ to $\Phi_{\varepsilon,\eta}(p)^{-1}(\{u = l(p) + t\})$ acquires Morse points with index equal to n (see [8] p. 30).

We have the lemma:

Lemma 9.2 *A topological space which becomes a space homotopically equivalent to a bouquet of real spheres of dimension n after attaching cells of dimension n is homotopically equivalent to a bouquet of real spheres of dimension $n-1$.*

Therefore, the space $\Phi_{\varepsilon,\eta}(p)^{-1}(\mathbb{D})$ is homotopically equivalent to a bouquet of real spheres of dimension $n-1$. Since this space retracts by deformation to the vanishing fiber

$$\Phi_{\varepsilon,\eta}(p)^{-1}((l(p) + t, f(p))),$$

this vanishing fiber is also homotopically equivalent to a bouquet of real spheres of dimension $n-1$.

Thus, we have proved that the Milnor fiber at p of the restriction of linear form to E has the homotopy type of a bouquet of real spheres of dimension $n-1$. This Milnor fiber is also one of the vanishing fibers at the point p. Since we can consider the other vanishing fibers to be Milnor fibers of the restriction of a general linear form to a lower dimensional hypersurface, we have proved the Theorem 9.1 for hypersurfaces.

9.3 Complete intersections

In the case of complete intersections, suppose that E is defined by $f_1 = \ldots = f_s = 0$. We may replace the equations $f_1, \ldots, f_s$ by general linear combinations of $f_1, \ldots, f_s$. Then, E is defined by f_s on the complete intersection $E_1 = \{f_1 = \ldots = f_{s-1} = 0\}$. By replacing the original equations by general linear combinations, the singular points of E_1 lie in the fixed points, i.e. in $E = \{f_1 = \ldots = f_s = 0\}$.

Therefore $E_1 \setminus E$ is non-singular and we can repeat a similar argument as the one we have developed above for hypersurfaces. The Lemma 9.2 will show that by attaching cells of dimension $\dim E_1 - 1$ the Milnor fiber of a general linear form restricted to E at p gives the Milnor fiber of a general linear form restricted to E_1 at p.

Since, by induction on the number of equations, we can suppose that the Milnor fiber of a general linear form restricted to E_1 at p has the homotopy type of a bouquet of real spheres of dimension $\dim E_1 - 1$, the Milnor fiber of a general linear form restricted to E at p has the homotopy type of a bouquet of spheres of dimension $\dim E - 1 = \dim E_1 - 2$.

We leave details to the reader.

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