# Topology and geometry of real singularities

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# Contents

# Chapter 1

# Introduction

This mini-course is aimed at young researchers and graduate students who want to learn basic tools and techniques of real singularity theory.

It starts with well-known notions and results of differential topology: Brouwer degree, index of a vector-field, Poincaré-Hopf theorem, Gauss-Bonnet theorem. Although theses notions may be very familiar to any confirmed researcher in singularity theory, we believe it is worth recalling them here.

Then in the next chapter, we apply these techniques of differential topology to some real analytic or semi-analytic sets and we get several nice formulas for topological invariants of these sets.

We end with a chapter about semi-algebraic sets. After a brief introduction on semi-algebraic sets and maps, we give several semi-algebraic singular versions of the Poincaré-Hopf theorem and the Gauss-Bonnet theorem.

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# Chapter 2

# Tools of differential topology

In this chapter, we give the main tools and results in differential topology that we will need and apply in the study of real singularities. Our main references are [GP], [Hi], [Mi1] and [Mi2].

## 2.1 The Brouwer degree

Let $M$ and $N$ be two oriented $n$-dimensional manifolds (without boundary) and let $f : M \to N$ be a smooth (i.e. $C^\infty$) map. We assume that $M$ is compact and $N$ is connected. Let $x$ be a regular point of $f$. This means that $Df(x) : T_xM \to T_{f(x)}N$ is a linear isomorphism between oriented vector spaces. We define the "sign" of $Df(x)$ to be $+1$ (resp. $-1$) if $Df(x)$ preserves (resp. reverses) the orientation.

**Definition 2.1.1** *For any regular value $y \in N$, we define:*

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign } Df(x).$$

**Remark 2.1.2** *Since $y$ is a regular value of $f$, $f^{-1}(y)$ consists of regular points. Since $M$ is compact, $f^{-1}(y)$ is a finite number of points since it is a 0-dimensional manifold.*

**Theorem 2.1.3** *The integer $\deg(f, y)$ does not depend on the choice of the regular value $y$.*

**Definition 2.1.4** *It is called the (Brouwer) degree of $f$, denoted by $\deg f$.*

**Theorem 2.1.5** *If $f$ is smoothly homotopic to $g$ then $\deg f = \deg g$.*

For the proof of these two theorems, the reader is referred to Milnor's book [Mi1].

Examples: 1) The map on the left has degree 0 and the map on the right has degree 1.

2) The map pictured below has degree 2.

3) Let $r_i : S^n \to S^n$ be defined by:

$$r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

The degree of $r_i$ is $-1$.

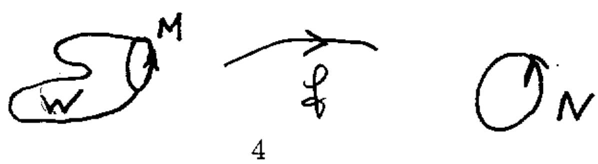
4) The map $S^1 \subset \mathbb{C} \to S^1 \subset \mathbb{C}$, $z \mapsto z^m$, $m \in \mathbb{Z}^*$ has degree $m$. Actually each point has exactly $m$ preimages and the map is regular and preserves orientation if $m > 0$ and reverses it if $m < 0$.

**Proposition 2.1.6** *If $M$, $N$ and $L$ are three smooth compact oriented manifolds of the same dimension and $f : M \to N$ and $g : N \to L$ are two smooth maps the* $\deg(g \circ f) = \deg g \times \deg f$.

*Proof.* It is easy from the definition. □

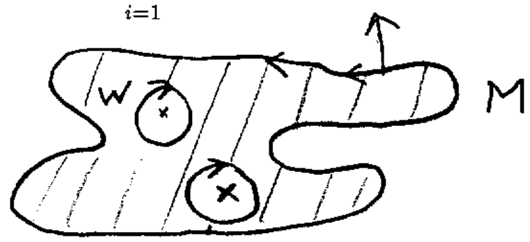
Example: The antipodal map $\sigma : S^n \to S^n$, $x \mapsto -x$ has degree $(-1)^{n+1}$ because $\sigma = r_1 \circ r_2 \circ \cdots \circ r_{n+1}$. Hence if $n$ is even, $\sigma$ is not smoothly homotopic to the identity.

**Theorem 2.1.7** *If $M = \partial W$ and $f : M \to N$ extends to $F : W \to N$ then* $\deg f = 0$.



Application: Assume that $M$ is a smooth compact connected hypersurface in $\mathbb{R}^n$. By the Jordan-Brouwer Separation Theorem, it bounds a connected open set $D \subset \mathbb{R}^n$, i.e. $M = \partial\overline{D}$. This induces a natural orientation on $M$. Let $F : \overline{D} \to \mathbb{R}^n$ be a map that does not vanish on $\partial\overline{D} = M$. We assume that inside $D$, $F$ has a finite number of zeroes $p_1, \ldots, p_m$ that are all regular points of $F$. We can define $\overline{F} = \frac{F}{|F|} : M \to S^{n-1}$. Then we have:

$$\deg \overline{F} = \sum_{i=1}^{m} \text{sign } \det[DF(p_i)].$$




Let us explain this equality. We remove a small open ball $B(p_i, \varepsilon_i)$ around each $p_i$. Let $W = \overline{D} \setminus \cup_{i=1}^{m} B(p_i, \varepsilon_i)$, it is a manifold with boundary:

$$\partial W = M \bigcup \cup_{i=1}^{m} S(p_i, \varepsilon_i).$$

The submanifold $\partial W$ is oriented by the canonical orientation of the boundary. Let us consider $\overline{F}_{\partial W} : \partial W \to S^{n-1}, x \mapsto \frac{F}{|F|}$. Then $\overline{F}_{\partial W}$ extends to $W$, so deg $\overline{F}_{\partial W} = 0$. But, on the other hand, we have:

$$\deg \overline{F}_{\partial W} = \deg \overline{F} - \sum_{i=1}^{m} \deg \overline{F}_i,$$

where $\overline{F}_i = \frac{F}{|F|} : S(p_i, \varepsilon_i) \to S^{n-1}$. A minus sign appears here because the orientation of $S(p_i, \varepsilon_i)$ as a component of the boundary of $W$ is the opposite of the orientation of $S(p_i, \varepsilon_i)$ as the boundary of $B(p_i, \varepsilon_i)$.

But, since $p_i$ is a regular point of $F$, the degree of $\overline{F}_i$ is equal to the sign of $\det[DF(p_i)]$, because $\frac{F}{|F|} : S(p_i, \varepsilon_i) \to S^{n-1}$ is homotopic to the map:

$$S(p_i, \varepsilon_i) \to S^{n-1}, p \mapsto \frac{DF(p_i)(p - p_i)}{|DF(p_i)(p - p_i)|},$$

and therefore has the same degree as the map:

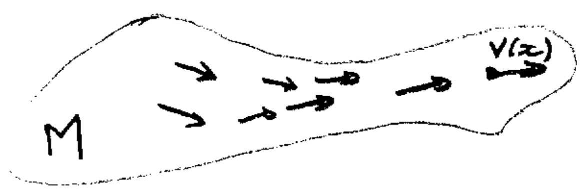
$$S(0, \varepsilon_i) \to S^{n-1}, h \mapsto \frac{DF(p_i)(h)}{|DF(p_i)(h)|}.$$

This last map has degree equal to sign det $[DF(p_i)]$, because the map $h \to DF(p_i)(h)$ is homotopic to $\pm Id_{\mathbb{R}^n}$, depending on the sign of $\det[DF(p_i)]$ since $GL(n, \mathbb{R})$ has two connected components.

## 2.2 Vector fields and Poincaré-Hopf theorem

**Definition 2.2.1** *Let $M$ be a smooth manifold. A vector field $V$ on $M$ is a smooth mapping $V : M \to TM$ such that for all $x \in M$, $pr(V(x)) \in T_xM$, where $pr : TM \to M$ is the natural projection.*

**Remark 2.2.2** *If $M \subset \mathbb{R}^n$ then a vector field is just a smooth mapping $V : M \to \mathbb{R}^n$ such that for all $x \in M$, $V(x) \in T_xM$.*




**Definition 2.2.3** *Let $p$ be an isolated zero of a vector field $V$ on a manifold $M$ of dimension $n$. In local coordinates, we can view $V$ as a mapping from a small open set $U \subset \mathbb{R}^n$ to a small open set $U' \subset \mathbb{R}^n$ where $0 \in U$ and $0 \in U'$ and such that $0$ is the only zero of $V$ in $U$. We define the Poincaré-Hopf index of $V$ at $p$ by:*

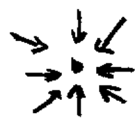
$$\text{Ind}(V, p) = \text{degree of } \frac{V}{|V|} : S_\varepsilon^{n-1} \to S^{n-1},$$

*where $S_\varepsilon^{n-1}$ is a small sphere included in $U$.*

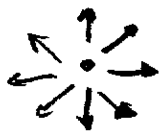
<u>Examples in $\mathbb{R}^2$:</u>

1. If $V(x, y) = (y, -x)$ (circulation) then $\text{Ind}(V, 0) = +1$.

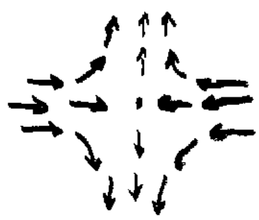
2. If $V(x,y) = (-x,-y)$ (sink) then $\mathrm{Ind}(V,0) = +1$.



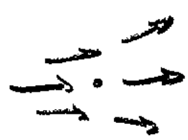
3. If $V(x,y) = (x,y)$ (source) then $\mathrm{Ind}(V,0) = +1$.



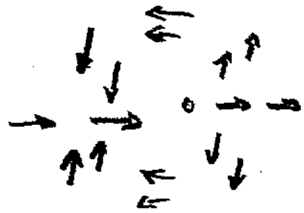
4. If $V(x,y) = (-x,y)$ (saddle) then $\mathrm{Ind}(V,0) = -1$.



5. If $V(x,y) = (x^2, x+y)$ then $\mathrm{Ind}(V,0) = 0$.



6. If $V(x,y) = (x^2 - y^2, 2xy)$ ($z \mapsto z^2$ in complex coordinates) then $\mathrm{Ind}(V,0) = +2$.



**Remark 2.2.4** *We have to check that this definition does not depend on the local coordinates (see [Mi1] and [GP] for example).*

**Theorem 2.2.5** *(Poincaré-Hopf theorem) Let $M$ be a smooth compact manifold. Let $V$ be a smooth vector field on $M$, with a finite number of zeroes $p_1, \dots, p_k$. Then we have:*

$$\chi(M) = \sum_{i=1}^{k} \mathrm{Ind}(V, p_i).$$

*Proof.* See [Mi1], [GP] or [Hi]. □

## 2.3 Morse functions

**Definition 2.3.1** *Let $M$ be a smooth manifold of dimension $n$, let $p \in M$ and $f : M \to \mathbb{R}$ be a smooth function. Let $(x_1, \dots, x_n)$ be local coordinates around $p$ in $M$. We say that $p$ is a non-degenerate critical point of $f$ if $p$ is a critical point of $f$ (i.e. $\frac{\partial f}{\partial x_1}(p) = \dots = \frac{\partial f}{\partial x_n}(p) = 0$) and the matrix:*

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{1 \le i,j \le n}$$

*is non-singular.*

**Remark 2.3.2** *It can be checked that this definition does not depend on the choice of the local coordinate system (see [GP]).*

**Proposition 2.3.3** *(Morse lemma) Let $p \in M$, $\dim M = n$, be a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$. There exists a local coordinate system $(u_1, \dots, u_n)$ around $p$ such that:*

$$f = f(p) - u_1^2 - \dots - u_\lambda^2 + u_{\lambda+1}^2 + \dots + u_n^2.$$

*Proof.* See [Mi2]. □

**Definition 2.3.4** *The integer $\lambda$ is called the index of $f$ at $p$.*

**Corollary 2.3.5** *Non-degenerate critical points are isolated (in the set of critical points).*

**Definition 2.3.6** *Let $M$ be a smooth manifold. A function $f : M \to \mathbb{R}$ is a Morse function if it admits only non-degenerate critical points.*

**Theorem 2.3.7** *(Openness and density) For any manifold $M$, Morse functions form a dense open set in $C_s^\infty(M,\mathbb{R})$ (Whitney strong topology).*

*Proof.* It is a consequence of Thom's transversality theorem (See [Hi], [GG] or [AGV]). $\square$

For our applications, we will be mainly interested in fibres of analytic or polynomial mappings, so from now on, we shall assume that $M \subset \mathbb{R}^N$ and that $\dim M = n$. Let $f : M \to \mathbb{R}$ be a smooth function. This defines a vector field $\nabla_M f$ (the gradient vector field of $f$ on $M$) by:

$$\forall p \in M, \forall v \in T_pM, Df(p)(v) = \langle \nabla_M f(p), v\rangle.$$

Hence $p$ is a critical point of $f$ if and only if $\nabla_M f(p) = 0$. If $p$ is a Morse critical point of $f$ of index $\lambda$, then there is a local coordinate system such that:

$$f = f(p) - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_n^2,$$

and so:

$$\nabla_M f = (-2u_1, \ldots, -2u_\lambda, 2u_{\lambda+1}, \ldots, 2u_n).$$

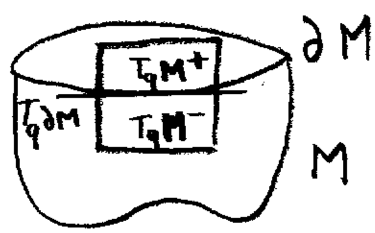
Hence the Poincaré-Hopf index $\mathrm{Ind}(\nabla_M f, p)$ is equal to $(-1)^\lambda$ because, as already explained above, the mapping $\frac{\nabla_M f}{|\nabla_M f|} : S(p,\varepsilon) \to S^{n-1}$ has degree equal to sign $\det[D(\nabla_M f)(p)]$. We can state:

**Theorem 2.3.8** *Let $M \subset \mathbb{R}^N$ be a smooth compact manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical points $p_1, \ldots, p_k$. Then, we have:*

$$\chi(M) = \sum_{i=1}^{k} (-1)^{\lambda(p_i)},$$

*where $\lambda(p_i)$ is the Morse index of $p_i$.*

We will also consider the case of manifolds with boundary. Let $(M, \partial M) \subset \mathbb{R}^N$ be a manifold with boundary. Let $q \in \partial M$, then $T_q\partial M$ is a hyperplane in $T_qM$ and $T_qM = T_q\partial M \sqcup T_qM^+ \sqcup T_qM^-$ where $T_qM^+$ consists of outwards pointing vectors (outward vectors for short) and $T_qM^-$ consists of inwards pointing vectors.



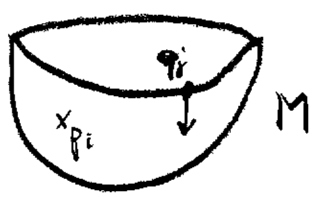
**Definition 2.3.9** *Let $q \in \partial M$ and let $f : (M, \partial M) \to \mathbb{R}$ be a smooth function. We say that $q$ is a correct critical point of $f$ if $q$ is a critical point of $f_{|\partial M} : \partial M \to \mathbb{R}$ and $Df(q)_{|T_q M}$ is not identically zero.*

**Definition 2.3.10** *We say that $f : (M, \partial M) \to \mathbb{R}$ is a correct Morse function if $f$ admits only Morse critical points on $M \setminus \partial M$ and $f_{|\partial M}$ admits only Morse correct critical points.*

**Theorem 2.3.11** *Let $(M, \partial M) \subset \mathbb{R}^N$ be a compact manifold with boundary and let $f : M \to \mathbb{R}$ be a correct Morse function. Denote by $p_1, \ldots, p_k$ the critical points of $f_{|M \setminus \partial M}$ and by $q_1, \ldots, q_l$ those of $f_{|\partial M}$. Then we have:*

$$\chi(M) = \sum_{i=1}^{k} (-1)^{\lambda(p_i)} + \sum_{j \mid \nabla_M f(q_j) \ inward} (-1)^{\mu(q_j)},$$

*where $\lambda(p_i)$ is the Morse index of $p_i$ and $\mu(q_j)$ is the Morse index of $q_j$.*




The following result is due to Haefliger [Ha] and Samelson [Sa].
Application: Let $M^n \subset \mathbb{R}^{n+1}$ be a compact hypersurface canonically oriented. Then $M$ is the boundary of a compact manifold $W$ of dimension $n+1$. Let $g : M \to S^n$ be the outwards pointing unit normal vector field. We have:

$$\deg g = \chi(W).$$


*Proof.* By Sard's theorem, we can find $a \in S^n$ such that $a$ and $-a$ are regular value of $g$. Let us write $\{q_1, \ldots, q_l\} = g^{-1}(\pm a)$. Let $a^* : \mathbb{R}^{n+1} \to \mathbb{R}$ be the function defined by $a^*(x) = \langle a, x \rangle$ and let us consider $a^*_{|M} : M \to \mathbb{R}$, $x \mapsto \langle a, x \rangle$ its restriction to $M$. The critical points of $a^*_{|M}$ are exactly the $q_j$'s and furthermore, by a determinant computation, there are non-degenerate. Hence $a^*_{|M}$ is a Morse function. It is straightforward to see that $a^*_{|W}$ is a correct Morse function. By the previous theorem, we know that:

$$\chi(W) = \sum_{j \ | \ \nabla a^*(q_j) \ inward} (-1)^{\mu(q_j)},$$

where $\mu(q_j)$ is the Morse index of $a^*_{|M}$ at $q_j$. But $\nabla a^* = a$ so $\nabla a^*(q)$ is inward if and only if $g(q) = -a$. Therefore,

$$\chi(W) = \sum_{j \ | \ g(q_j)=-a} (-1)^{\mu(q_j)}.$$

It remains to relate $(-1)^{\mu(q_j)}$ to the local degree of $q$ at $q_j$. We use the following lemma.

**Lemma 2.3.12** *We have:* $\deg(g, q_j) = (-1)^n \mathrm{sign}\langle g(q_j), a\rangle^n (-1)^{\mu(q_j)}$.

*Proof.* See [Du6], Lemma 2.3. □

If $g(q_j) = -a$ then

$$\deg(g, q_j) = (-1)^n \left(\mathrm{sign}(-|a|^2)\right)^n (-1)^{\mu(q)} = (-1)^{\mu(q)}.$$

Finally, we find that:

$$\chi(W) = \sum_{j \ | \ g(q_j)=-a} \deg(g, q_j) = \deg\, g.$$

□

In the following chapters of this mini-course, we will use relative versions of the two previous theorems on Morse theory.

**Theorem 2.3.13** *Let $M \subset \mathbb{R}^N$ be a smooth compact manifold and let $f : M \to \mathbb{R}$ be a Morse function with critical points $p_1, \ldots, p_k$. For any $\alpha \in \mathbb{R}$, we have:*

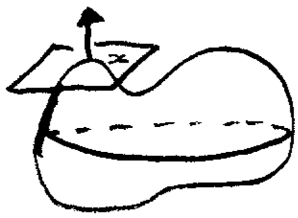
$$\chi(M, M \cap \{f \geq \alpha\}) = \sum_{i \ | \ f(p_i) > \alpha} (-1)^{\lambda(p_i)},$$

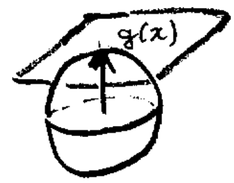
*where $\lambda(p_i)$ is the Morse index of $p_i$.*

Theorem 2.3.11 has a similar relative version.

## 2.4 The Gauss-Bonnet theorem

Let $M \subset \mathbb{R}^{n+1}$ be a compact hypersurface, canonically oriented as the boundary of a compact manifold with boundary $W$. Let $g : M \to S^n$ be the Gauss map. Its Jacobian $J_g(x) = k(x)$ is called the curvature of $M$ at $x$. It is the determinant of the differential $Dg(x) : T_xM \to T_{g(x)}S^n = T_xM$.




**Theorem 2.4.1** *We have:*

$$\int_M k(x)dx = \mathrm{vol}(S^n)\chi(W).$$

*Proof.* We denote by $dv$ the volume form on $S^n$. By integral calculus on manifolds, we can write:

$$\int_M k(x)dx = \int_M J_g(x)dx = \int_M g^*(dv) = \deg\, g \int_{S^n} dv = \deg\, g \times \mathrm{vol}(S^n).$$

But we know that $\deg\, g = \chi(W)$. □

The following corollary is due to Hopf [Ho].

**Corollary 2.4.2** *If $M$ is even-dimensional, we have:*

$$\int_M k(x)dx = \frac{1}{2}\mathrm{vol}(S^n)\chi(M).$$

*Proof.* Use the equality $\chi(M) = 2\chi(W)$. □

# Chapter 3

# The Eisenbud-Levine formula, the Khimshiashvili formula and applications

## 3.1 The Eisenbud-Levine formula

As seen in the first chapter, the Poincaré-Hopf index of a vector field plays an important role in the topology of manifolds. Here we present an algebraic formula for this index.

Let $f = (f_1, \ldots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a $C^\infty$ map-germ (this is exactly the local expression of a vector field on a smooth manifold). We assume that:

$$Q(f) = \frac{C^\infty(\mathbb{R}^n, 0)}{(f_1, \ldots, f_n)},$$

is a finite dimensional vector space over $\mathbb{R}$. Here $C^\infty(\mathbb{R}^n, 0)$ is the algebra of germs at $0 \in \mathbb{R}^n$ of $C^\infty$ real valued functions and $(f_1, \ldots, f_n)$ is the ideal generated by the components $f_1, \ldots, f_n$ of $f$. We write $\dim_{\mathbb{R}} Q(f) < +\infty$. We denote by $J_f$ the jacobian of the map-germ $f$. Namely, we have:

$$J_f = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}.$$

**Theorem 3.1.1** *(The Eisenbud-Levine formula) Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a $C^\infty$ map-germ such that $\dim_{\mathbb{R}} Q(f) < +\infty$. Then we have:*

1. *$0$ is isolated in $f^{-1}(0)$,*
2. *$J_f \neq 0$ in $Q(f)$,*

3. *$\forall g \in Q(f)$, $gJ_f = g(0)J_f$ in $Q(f)$,*

4. *let $\varphi : Q(f) \to \mathbb{R}$ be a linear form such that $\varphi(J_f) > 0$ and let $\Phi : Q(f) \times Q(f) \to \mathbb{R}$ be the bilinear symmetric form defined by $\Phi(g,h) = \varphi(gh)$ then $\Phi$ is non-degenerate and* signature $\Phi = \mathrm{Ind}(f,0)$.

*Proof.* See [EL], [AGV] or [BCRS]. For a first approach, see [Ei]. $\square$

Example: Let $f$ be the map-germ defined by:

$$\begin{array}{rcccc} f & : & (\mathbb{R}^2,0) & \to & (\mathbb{R}^2,0) \\ & & (x,y) & \mapsto & (x^2-y^2,2xy). \end{array}$$

We have: $Q(f) = \frac{C^\infty(\mathbb{R}^2,0)}{(x^2-y^2,2xy)}$. We see that $\dim_{\mathbb{R}} Q(f) = 4$ and that $\bar{1}$, $\bar{x}$, $\bar{y}$ and $\overline{x^2+y^2}$ form a basis of $Q(f)$. It is clear that 0 is isolated in $f^{-1}(0)$. Let us compute $J_f$:

$$J_f(x,y) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2).$$

Let $\varphi : Q(f) \to \mathbb{R}$ be the linear form given by:

$$\varphi(\bar{1}) = \varphi(\bar{x}) = \varphi(\bar{y}) = 0 \text{ and } \varphi(\overline{x^2+y^2}) = \frac{1}{4}.$$

Then $\varphi(J_f) = 1$. Let $\Phi$ be the linear symmetric form defined by $\Phi(P,Q) = \varphi(PQ)$. Let us compute its matrix in the basis $(\bar{1}, \bar{x}, \bar{y}, \overline{x^2+y^2})$. We have:

$$\Phi(\bar{1},\bar{1}) = \varphi(\bar{1}) = 0, \Phi(\bar{1},\bar{x}) = \Phi(\bar{x},\bar{1}) = 0, \Phi(\bar{1},\bar{y}) = \Phi(\bar{y},\bar{1}) = 0,$$

$$\Phi(\bar{x},\bar{x}) = \varphi(\bar{x}^2) = \varphi(\frac{1}{2}\overline{x^2+y^2}) = \frac{1}{8},$$

$$\Phi(\bar{y},\bar{y}) = \varphi(\bar{y}^2) = \varphi(\frac{1}{2}\overline{x^2+y^2}) = \frac{1}{8},$$

$$\Phi(\bar{x},\bar{y}) = \varphi(\overline{xy}) = \varphi(\bar{0}) = 0 = \Phi(\bar{y},\bar{x}),$$

$$\Phi(\bar{1},\overline{x^2+y^2}) = \frac{1}{4}, \Phi(\bar{x},\overline{x^2+y^2}) = \varphi(\bar{x}^3 + \overline{xy^2}) = \varphi(\bar{0}) = 0,$$

$$\Phi(\bar{y},\overline{x^2+y^2}) = 0, \Phi(\overline{x^2+y^2},\overline{x^2+y^2}) = \varphi\big(\overline{(x^2+y^2)(x^2+y^2)}\big) = 0.$$

So this matrix is:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\frac{1}{8}$ with multiplicity 2, $\frac{1}{4}$ with multplicity 1 and $-\frac{1}{4}$ with multiplicity 1. So the signature of $\Phi$ is $3 - 1 = 2 = \text{Ind}(f, 0)$.

The Eisenbud-Levine formula gives an algebraic formula for the index of a vector field, hence an algebraic and "effective" way to compute a topological data. In the sequel, using technics introduced in the first chapter, we will present several formulas relating topological invariants to indices of vector fields. Thanks to the Eisenbud-Levine formula, these topological invariants become algebraically computable.

## 3.2 The Khimshiashvili formula

From now on, we will restrict ourselves to the analytic or polynomial case.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic-map germ with an isolated critical point at 0. The Khimshiashvili formula (see [Kh]) relates the Poincaré-Hopf index of the gradient vector field $\nabla f$ of $f$ to the topology of a small regular level of $f$.

**Theorem 3.2.1** *We have:*

$$\chi\big(f^{-1}(\delta) \cap B_\varepsilon^n\big) = 1 - \text{sign}(-\delta)^n \text{Ind}(\nabla f, 0), \qquad (1)$$

*where $\delta$ is a regular value of $f$ and $0 < |\delta| \ll \varepsilon \ll 1$, and:*

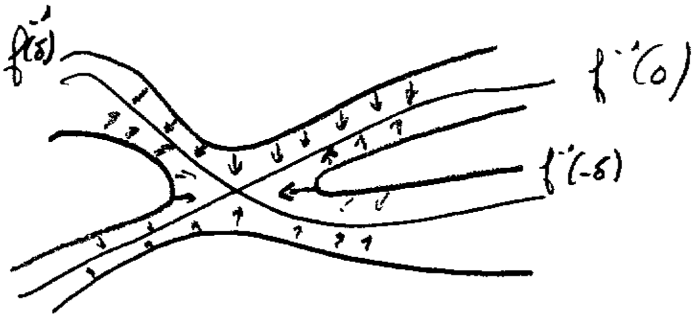
$$\chi(\{f \geq \delta\} \cap B_\varepsilon^n) - \chi(\{f \leq \delta\} \cap B_\varepsilon^n) = \text{sign}(-\delta)^{n+1} \text{Ind}(\nabla f, 0). \qquad (2)$$

*Proof.* Let $U$ be a small open set of $\mathbb{R}^n$ such that $0 \in U$, and $f$ is defined in $U$. We pertub $f$ in a Morse function $\tilde{f} : U \to \mathbb{R}$. Let $p_1, \ldots, p_k$ be the critical points of $\tilde{f}$, with respective indices $\lambda_1, \ldots, \lambda_k$. Let $\delta > 0$, by Morse theory we have:

$$\chi\big(f^{-1}([-\delta, \delta]) \cap B_\varepsilon^n\big) - \chi\big(f^{-1}(-\delta) \cap B_\varepsilon^n\big) = \sum_{i=1}^{k} (-1)^{\lambda_i}.$$

Actually we can choose $\tilde{f}$ sufficiently close to $f$ so that the $p_i$'s lie in $f^{-1}([-\frac{\delta}{4}, \frac{\delta}{4}])$. Now, $f^{-1}([-\delta, \delta]) \cap B_\varepsilon^n$ retracts to the central fibre $f^{-1}(0) \cap B_\varepsilon^n$ and $f^{-1}(0) \cap B_\varepsilon$ is the cone over $f^{-1}(0) \cap S_\varepsilon^{n-1}$) (see [Mi3]) so:

$$\chi\big(f^{-1}([-\delta, \delta]) \cap B_\varepsilon^n\big) = 1.$$




Moreover, we have:

$$\sum_{i=1}^{k}(-1)^{\lambda_i} = \sum_{i=1}^{k}\text{sign}\ \det[D(\nabla\tilde{f})(p_i)].$$

The sum on the right hand-side is the degree of the map $\frac{\nabla\tilde{f}}{|\nabla\tilde{f}|} : S_\varepsilon^{n-1} \to S^{n-1}$ which is equal, by homotopy, to the degree of $\frac{\nabla f}{|\nabla f|} : S_\varepsilon^{n-1} \to S^{n-1}$. By definition, this last degree is $\text{Ind}(\nabla f, 0)$. This gives the result for a negative regular value. For a positive regular value, we apply the result to $-f$ and use the relation $\text{Ind}(-\nabla f, 0) = (-1)^n\text{Ind}(\nabla f, 0)$. This proves formula (1). Formula (2) is proved with similar arguments. □

We will call $f^{-1}(\delta) \cap B_\varepsilon^n$ the (positive or negative) real Milnor fibre. The following formulas are due to Arnol'd [Ar] and Wall [Wa].

**Corollary 3.2.2** *With the same hypothesis on $f$, we have:*

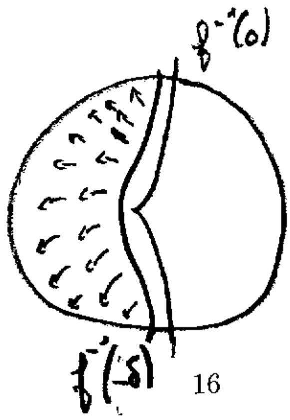
$$\chi(\{f \leq 0\} \cap S_\varepsilon^{n-1}) = 1 - \text{Ind}(\nabla f, 0),$$

$$\chi(\{f \geq 0\} \cap S_\varepsilon^{n-1}) = 1 + (-1)^{n-1}\text{Ind}(\nabla f, 0).$$

*If $n$ is even, we have:*

$$\chi(\{f = 0\} \cap S_\varepsilon^{n-1}) = 2 - 2\ \text{Ind}(\nabla f, 0).$$

*Proof.* By a deformation argument due to Milnor [Mi3], $f(-\delta) \cap B_\varepsilon^n$, $\delta > 0$, is homeomorphic to $\{f \leq -\delta\} \cap S_\varepsilon^{n-1}$, which is homeomorphic to $\{f \leq 0\} \cap S_\varepsilon^{n-1}$ if $\delta$ is very small.




□

## 3.3 The Fukui formula

The above real Milnor fibre can be also written $f_t^{-1}(0)\cap B_\varepsilon^n$, $0<|t|\ll\varepsilon\ll 1$, where $f_t(x)=f(x)-t$. In this section, we will present a method for the computation of the Euler characteristic of $f_t^{-1}(0)\cap B_\varepsilon^n$, $0<|t|\ll\varepsilon\ll 1$, where $f_t$ is a one-parameter deformation of $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$. It is interesting to study such deformations because the topology of the fibre is somehow richer than the one of the real Milnor fibre and contains more information about the singularity. The setting is described below.

Let $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ be an analytic function-germ with an isolated critical point at 0. Let $F:(\mathbb{R}^{n+1},0)\to(\mathbb{R},0)$, $(t,x)\mapsto F(t,x)$ be a one-parameter deformation of $f$, i.e. $F_0(x)=F(0,x)=f(x)$. Let $H:(\mathbb{R}^{n+1},0)\to(\mathbb{R}^{n+1},0)$ be defined by $H(t,x)=(F,\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n})$. We assume that $H$ has an isolated critical zero at 0, so that $\mathrm{Ind}(H,0)$ is well-defined.

**Lemma 3.3.1** *The function $F$ has an isolated critical point at the origin.*

*Proof.* By the Curve Selection Lemma [Mi3], $\nabla F^{-1}(0)$ is included in $F^{-1}(0)$. Hence $\nabla F^{-1}(0)\subset H^{-1}(0)$. □

**Lemma 3.3.2** *For $t\neq 0$ small, the fibre $f_t^{-1}(0)=\{x\in\mathbb{R}^n\mid F(t,x)=0\}$ is smooth in a neighborhood of the origin.*

*Proof.* The point $x\in\mathbb{R}^n$ is a critical point of $f_t^{-1}(0)$ if and only if $\forall i\in\{1,\ldots,n\}$, $\frac{\partial f_t}{\partial x_i}(x)=0$. This implies that $\frac{\partial F}{\partial x_i}(t,x)=0$ and that $H(t,x)=0$. □

The following theorem was proved by Fukui [Fuk].

**Theorem 3.3.3** *For $0<|\delta|\ll\varepsilon\ll 1$, we have:*

$$\chi\big(F^{-1}(\delta)\cap\{t\geq 0\}\cap B_\varepsilon^{n+1}\big)-\chi\big(F^{-1}(\delta)\cap\{t\leq 0\}\cap B_\varepsilon^{n+1}\big)=\\ -\mathrm{sign}(-\delta)^{n+1}\mathrm{Ind}(H,0).$$

*Proof.* We work in a small open set $U$ of $\mathbb{R}^{n+1}$ that contains 0. Let $\pi:\mathbb{R}^n\to\mathbb{R},(t,x)\mapsto t$ be the projection on the first coordinate. After a small perturbation of $F$, we can assume that $\pi_{|F^{-1}(\delta)\cap\mathring{B}_\varepsilon^n}:F^{-1}(\delta)\cap\mathring{B}_\varepsilon^n\to\mathbb{R}$ admits only Morse critical points $p_1,\ldots,p_k$ with respective indices $\lambda_1,\ldots,\lambda_k$. By Morse theory, we have:

$$\chi\big(F^{-1}(\delta)\cap\{t\geq 0\}\cap B_\varepsilon^{n+1}\big)-\chi\big(F^{-1}(\delta)\cap\{t=0\}\cap B_\varepsilon^{n+1}\big)=$$

$$\sum_{j \mid \pi(p_j)>0} (-1)^{\lambda_j},$$

$$\chi\big(F^{-1}(\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1}\big) - \chi\big(F^{-1}(\delta) \cap \{t = 0\} \cap B_\varepsilon^{n+1}\big) = (-1)^n \sum_{j \mid \pi(p_j)<0} (-1)^{\lambda_j}.$$

Here we have to notice that $F^{-1}(\delta) \cap B_\varepsilon^{n+1}$ is a manifold with boundary. But, by the Curve Selection Lemma, we can prove that the critical points of $\pi_{|F^{-1}(\delta)\cap S_\varepsilon^n}$ in $\{t \geq 0\}$ point outwards, hence they do not appear in the above equality. Now it is easy to see that the critical points of $\pi_{|F^{-1}(\delta)\cap \mathring{B}_\varepsilon^{n+1}}$ are exactly the zeros of $H_\delta = (F - \delta, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$. A determinant computation shows that the $p_j$'s are non-degenerate zeros of $H_\delta$ and that:

$$(-1)^{\lambda_j} = -\text{sign } \pi(p_j)^{n+1} \times \text{ sign}(-\delta)^{n+1} \times \text{sign det}[DH_\delta(p_j)].$$

Thus we obtain:

$$\begin{aligned}\chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1} \cap \{t \geq 0\}\big) - \chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1} \cap \{\pi \leq 0\}\big) &= \\ -\sum_{j=1}^{k} \text{sign } \pi(p_j)^{n+1} \times (-1)^{\lambda_j} &= \\ -\text{sign}(-\delta)^{n+1} \sum_{j=1}^{k} \text{sign det}[DH_\delta(p_j)] &= \\ -\text{sign}(-\delta)^{n+1} \left(\text{degree of } \tfrac{H_\delta}{|H_\delta|} : S_\varepsilon^n \to S^n\right) &= \\ -\text{sign}(-\delta)^{n+1} \text{Ind}(H, 0).\end{aligned}$$

The last equality is explained by the fact that the maps $\frac{H_\delta}{|H_\delta|}$ and $\frac{H}{|H|}$ are homotopic if $\delta$ is small enough. □

**Corollary 3.3.4** *If $n$ is even, then we have:*

$$\chi\big(f_t^{-1}(0) \cap B_\varepsilon^n\big) = 1 - \text{Ind}(\nabla f, 0),$$

$$\chi(\{f_t \geq 0\} \cap B_\varepsilon^n) - \chi(\{f_t \leq 0\} \cap B_\varepsilon^n) = \text{Ind}(\nabla F, 0) + \text{sign}(t) \text{ Ind}(H, 0).$$

*If $n$ is odd, then we have:*

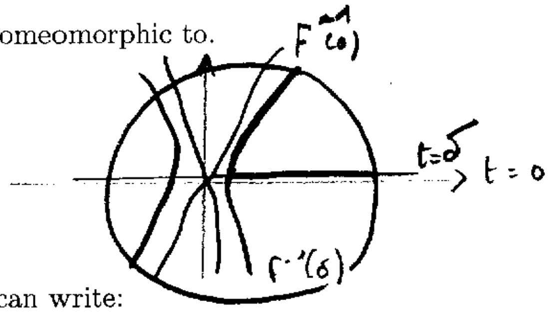
$$\chi\big(f_t^{-1}(0) \cap B_\varepsilon^n\big) = 1 - \text{Ind}(\nabla F, 0) - \text{sign}(t) \text{ Ind}(H, 0),$$

$$\chi(\{f_t \geq 0\} \cap B_\varepsilon^n) - \chi(\{f_t \leq 0\} \cap B_\varepsilon^n) = \text{Ind}(\nabla f, 0).$$

*Proof.* By a deformation argument, we have for $\delta > 0$:

$$F^{-1}(\delta)\cap\{t \geq 0\}\cap B_\varepsilon^{n+1} \simeq \{F \geq 0\}\cap\{t \geq 0\}\cap S_\varepsilon^n \simeq \{F \geq 0\}\cap\{t = \delta\}\cap B_\varepsilon^{n+1},$$

where $\simeq$ means homeomorphic to.




Similarly, we can write:

$$F^{-1}(\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1} \simeq \{F \geq 0\} \cap \{t = -\delta\} \cap B_\varepsilon^{n+1},$$

$$F^{-1}(-\delta) \cap \{t \geq 0\} \cap B_\varepsilon^{n+1} \simeq \{F \leq 0\} \cap \{t = \delta\} \cap B_\varepsilon^{n+1},$$

$$F^{-1}(-\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1} \simeq \{F \leq 0\} \cap \{t = -\delta\} \cap B_\varepsilon^{n+1}.$$

By Khimshiashvili's formula, we get:

$$\chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1}\big) = 1 + (-1)^n \mathrm{Ind}(\nabla F, 0),$$

$$\chi\big(F^{-1}(-\delta) \cap B_\varepsilon^{n+1}\big) = 1 - \mathrm{Ind}(\nabla F, 0),$$

$$\chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1} \cap \{t = 0\}\big) = 1 + (-1)^{n-1} \mathrm{Ind}(\nabla f, 0),$$

$$\chi\big(F^{-1}(-\delta) \cap B_\varepsilon^{n+1} \cap \{t = 0\}\big) = 1 - \mathrm{Ind}(\nabla f, 0).$$

By the Mayer-Vietoris sequence, we have:

$$\chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1}\big) + \chi\big(F^{-1}(\delta) \cap B_\varepsilon^{n+1} \cap \{t = 0\}\big) =$$

$$\chi\big(F^{-1}(\delta) \cap \{t \geq 0\} \cap B_\varepsilon^{n+1}\big) + \chi\big(F^{-1}(\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1}\big) =$$

$$2 + (-1)^n \mathrm{Ind}(\nabla F, 0) + (-1)^{n-1} \mathrm{Ind}(\nabla f, 0),$$

and:

$$\chi\big(F^{-1}(-\delta) \cap B_\varepsilon^{n+1}\big) + \chi\big(F^{-1}(-\delta) \cap B_\varepsilon^{n+1} \cap \{t = 0\}\big) =$$

$$\chi\big(F^{-1}(-\delta) \cap \{t \geq 0\} \cap B_\varepsilon^{n+1}\big) + \chi\big(F^{-1}(-\delta) \cap \{t \leq 0\} \cap B_\varepsilon^{n+1}\big) =$$

$$2 - \mathrm{Ind}(\nabla F, 0) - \mathrm{Ind}(\nabla f, 0).$$

Applying the previous theorem, we get:

$$\chi\big(\{F \geq 0\} \cap \{t = \delta\} \cap B_\varepsilon^{n+1}\big) = \chi\big(F^{-1}(\delta) \cap \{t \geq 0\} \cap B_\varepsilon^{n+1}\big) =$$

$$1 + \frac{(-1)^n}{2}\big(\mathrm{Ind}(\nabla F, 0) - \mathrm{Ind}(\nabla f, 0) + \mathrm{Ind}(H, 0)\big).$$

Similarly, we have:

$$\chi\big(\{F \geq 0\} \cap \{t = -\delta\} \cap B_\varepsilon^{n+1}\big) = \\ 1 + \frac{(-1)^n}{2}\big(\mathrm{Ind}(\nabla F, 0) - \mathrm{Ind}(\nabla f, 0) - \mathrm{Ind}(H, 0)\big),$$

$$\chi\big(\{F \leq 0\} \cap \{t = \delta\} \cap B_\varepsilon^{n+1}\big) = \\ 1 - \frac{1}{2}\big(\mathrm{Ind}(\nabla F, 0) + \mathrm{Ind}(\nabla f, 0) + \mathrm{Ind}(H, 0)\big),$$

$$\chi\big(\{F \leq 0\} \cap \{t = -\delta\} \cap B_\varepsilon^{n+1}\big) = \\ 1 - \frac{1}{2}\big(\mathrm{Ind}(\nabla F, 0) + \mathrm{Ind}(\nabla f, 0) - \mathrm{Ind}(H, 0)\big).$$

We conclude with:

$$\chi(B_\varepsilon^n) = 1 = \chi(\{f_t \geq 0\} \cap B_\varepsilon^n) + \chi(\{f_t \leq 0\} \cap B_\varepsilon^n) - \chi(\{f_t = 0\} \cap B_\varepsilon^n).$$

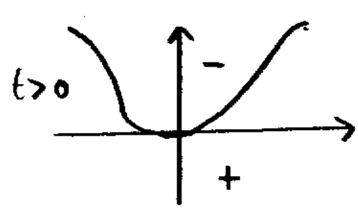
□

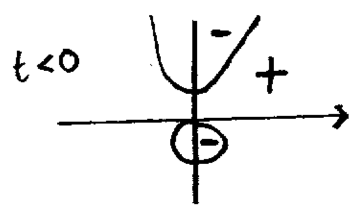
Example: Let $f(x,y) = x^2 - y^3$ and let $F(t,x,y) = x^2 - y^3 - ty$. If $t > 0$ then:

$$\chi\big(f_t^{-1}(0) \cap B_\varepsilon^n\big) = 1 \text{ and } \chi(\{f_t \geq 0\} \cap B_\varepsilon^n) - \chi(\{f_t \leq 0\} \cap B_\varepsilon^n) = 0.$$

If $t < 0$ then:

$$\chi\big(f_t^{-1}(0) \cap B_\varepsilon^n\big) = 1 \text{ and } \chi(\{f_t \geq 0\} \cap B_\varepsilon^n) - \chi(\{f_t \leq 0\} \cap B_\varepsilon^n) = -2.$$




Let us check that the above formulas hold in this example. We have $\nabla f(x,y) = (2x, -3y^2)$, hence $\mathrm{Ind}(\nabla f, 0) = 0$. Let us compute $\mathrm{Ind}(\nabla F, 0)$. We have $\nabla F(t,x,y) = (-y, 2x, -3y^2 - t)$. The matrix of the differential of $\nabla F$ is:

$$D(\nabla F)(t,x,y) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -6y \end{pmatrix}.$$

It is easy to see that $\nabla F^{-1}(0,0,\varepsilon) = (-\varepsilon, 0, 0)$ and that

$$\det[D(\nabla F)(-\varepsilon,0,0)] < 0.$$

This implies that $\mathrm{Ind}(\nabla F, 0) = -1$. Let us compute now $\mathrm{Ind}(H,0)$. We have $H(t,x,y) = (x^2 - y^3 - ty, 2x, -3y^2 - t)$ and

$$DH(t,x,y) = \begin{pmatrix} -y & 2x & -3y^2 - t \\ 0 & 2 & 0 \\ -1 & 0 & -6y \end{pmatrix}.$$

Let us search the preimages of $(0,0,\varepsilon)$ where $\varepsilon > 0$. If $(t,x,y)$ is such a preimage, then $x = 0$, $y^3 + ty = y(y^2+t) = 0$ and $3y^2 + t = -\varepsilon$. If $y^2 + t = 0$ then $3y^2 - y^2 = 2y^2 = -\varepsilon$, which is impossible. Therefore $y = 0$ and $t = -\varepsilon$. It is easy to see that $DH(-\varepsilon, 0, 0) = 2\varepsilon > 0$. We conclude that $\mathrm{Ind}(H,0) = +1$. Applying Fukui's formula, we recover the above values for $\chi(f_t^{-1}(0) \cap B_\varepsilon^n)$ and $\chi(\{f_t \geq 0\} \cap B_\varepsilon^n) - \chi(\{f_t \leq 0\} \cap B_\varepsilon^n)$.

## 3.4 Applications to map-germs with an isolated critical point

Let $\psi = (f_1, \ldots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$, $2 \leq k \leq n$, be a real analytic map-germ with an isolated critical point at the origin. This means that 0 is isolated in

$$\Sigma(\psi) = \{x \in \mathbb{R}^n \mid \mathrm{rank}(\nabla f_1(x), \ldots, \nabla f_k(x)) < k\}.$$

This implies that for any $l \in \{1, \ldots, k\}$ and for any $l$-tuple $(i_1, \ldots, i_l)$ of pairwise distinct elements of $\{1, \ldots, k\}$, the mapping $(f_{i_1}, \ldots, f_{i_l}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^l, 0)$ has an isolated critical point at the origin as well, for otherwise 0 would not be isolated in $\Sigma(\psi)$. Let $\phi = (f_1, \ldots, f_{k-1}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^{k-1}, 0)$. The following result was proved by Araújo dos Santos, Dreibelbis and the author [ADD].

**Proposition 3.4.1** *For $0 \ll |\delta| \ll \varepsilon \ll 1$, the following holds:*

*(i) If $n$ is even, we have:*

$$\chi(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n) = 1 - \mathrm{Ind}(\nabla f_1, 0);$$

*(ii) If $n$ is odd, we have* $\mathrm{Ind}(\nabla f_1, 0) = 0$ *and:*

$$\chi(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n) = 1.$$

*Proof.* Applying Morse theory for manifolds with boundary to $f_{k|\phi^{-1}(\delta)\cap B_\varepsilon^n}$, we have:

$$\chi\big(\phi^{-1}(\delta) \cap \{f_k \geq 0\} \cap B_\varepsilon^n\big) - \chi\big(\phi^{-1}(\delta) \cap f_k^{-1}(0) \cap B_\varepsilon^n\big) = 0,$$

because $f_{k|\phi^{-1}(\delta)}$ has no critical points for $\Sigma(\psi) = \{0\}$ and because the gradient vector field $\nabla f_{k|\phi^{-1}(\delta)\cap B_\varepsilon^n}$ points outwards at the critical points of $f_{k|\phi^{-1}(\delta)\cap S_\varepsilon^{n-1}}$ lying in $\{f_k > 0\}$. Similarly, we have:

$$\chi\big(\phi^{-1}(\delta) \cap \{f_k \leq 0\} \cap B_\varepsilon^n\big) - \chi\big(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n\big) = 0.$$

Summing these two equalities and using the Mayer-Vietoris sequence, we obtain that:

$$\chi\big(\phi^{-1}(\delta) \cap B_\varepsilon^n\big) = \chi\big(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n\big).$$

Applying this procedure $k-1$ times, we obtain that:

$$\chi\big(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n\big) = \chi\big(f_1{}^{-1}(\alpha_1) \cap B_\varepsilon^n\big),$$

where $\alpha_1$ is a small regular value of $f_1$.

By Khimshiashvili's formula, we know that:

$$\chi\big(f_1{}^{-1}(\alpha_1) \cap B_\varepsilon^n\big) = 1 - \mathrm{sign}(-\alpha_1)^n \mathrm{Ind}(\nabla f_1, 0).$$

Hence, if $n$ is even, we find that:

$$\chi\big(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n\big) = 1 - \mathrm{Ind}(\nabla f_1, 0).$$

If $n$ is odd, just changing $\alpha_1$ by $-\alpha_1$, we get that $\mathrm{Ind}(\nabla f_1, 0) = 0$ and $\chi\big(\phi^{-1}(\delta) \cap f_k{}^{-1}(0) \cap B_\varepsilon^n\big) = 1$. □

**Corollary 3.4.2** *Let $\gamma$ be a small regular value of $\psi$. If $n$ is even, we have:*

$$\chi\big(\psi^{-1}(\gamma) \cap B_\varepsilon^n\big) = 1 - \mathrm{Ind}(\nabla f_1, 0) = \cdots = 1 - \mathrm{Ind}(\nabla f_k, 0),$$

*and* $\mathrm{Ind}(\nabla f_1, 0) = \cdots = \mathrm{Ind}(\nabla f_k, 0)$. *If $n$ is odd, we have* $\mathrm{Ind}(\nabla f_1, 0) = \cdots = \mathrm{Ind}(\nabla f_k, 0)$ *and* $\chi\big(\psi^{-1}(\gamma) \cap B_\varepsilon^n\big) = 1$.

## 3.5 Real versions of the Lê-Greuel formula

In the previous sections, we studied map-germs from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^k, 0)$ when $k \in \{1, n\}$ or when $2 \le k \le n$ and the map-germ has an isolated critical point at the origin. Here we will investigate the general case.

Let $1 \le k < n$ and let $f = (f_1, \dots, f_k) : (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be an analytic map-germ such that 0 is an isolated singular point of $f^{-1}(0)$. This means that 0 is isolated in $\{x \in \mathbb{R}^n \mid \text{rank}[Df(x)] < k\} \cap f^{-1}(0)$. Let $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function-germ. Let $I$ be the ideal in $\mathcal{O}_{\mathbb{R}^n,0}$, the algebra of analytic function-germs at the origin, generated by $f_1, \dots, f_k$ and the minors $\frac{\partial(f_1,\dots,f_k,g)}{\partial(x_{i_1},\dots,x_{i_{k+1}})}$ and let $A_{\mathbb{R}} = \frac{\mathcal{O}_{\mathbb{R}^n,0}}{I}$. We will denote by $C_\delta^\varepsilon$ the real Milnor fibre $f^{-1}(\delta) \cap B_\varepsilon^n$, $0 < |\delta| \ll \varepsilon \ll 1$.

The following theorem appeared in [Du5].

**Theorem 3.5.1** *If* $\dim_{\mathbb{R}} A_{\mathbb{R}} < +\infty$ *then we have:*

$$\chi(C_\delta^\varepsilon \cap \{g \ge \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \le \alpha\}) \equiv$$

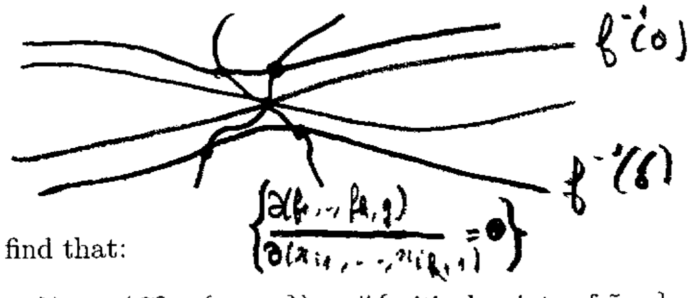
$$\chi(C_\delta^\varepsilon) - \chi(C_\delta^\varepsilon \cap \{g = \alpha\}) \equiv \dim_{\mathbb{R}} A_{\mathbb{R}} \bmod 2,$$

*where* $(\delta, \alpha)$ *is a regular value of* $(f, g)$ *such that* $0 \le |\delta| \ll |\varepsilon| \ll 1$.

*Proof.* We pertub $g$ in $\tilde{g}$ such that $\tilde{g}_{|C_\delta^\varepsilon}$ is Morse. By Morse theory, we have:

$$\chi(C_\delta^\varepsilon \cap \{g \ge \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g = \alpha\}) = \\ \#\{\text{critical points of } \tilde{g}_{|C_\delta^\varepsilon} \text{ such that } \tilde{g} > \alpha\} \bmod 2,$$

$$\chi(C_\delta^\varepsilon \cap \{g \le \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g = \alpha\}) = \\ \#\{\text{critical points of } \tilde{g}_{|C_\delta^\varepsilon} \text{ such that } \tilde{g} < \alpha\} \bmod 2.$$




Hence we find that:

$$\chi(C_\delta^\varepsilon \cap \{g \ge \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \le \alpha\}) = \#\{\text{critical points of } \tilde{g}_{|C_\delta^\varepsilon}\} \bmod 2.$$

By intersection theory, the right-hand side of this last equality is equal to $\dim_{\mathbb{R}} A_{\mathbb{R}} \bmod 2$. □

**Remark 3.5.2** *In the complex case, the Lê-Greuel formula ([Gr], [Le]) states that:*

$$\mu(f) + \mu(f,g) = \dim_{\mathbb{C}} A_{\mathbb{C}},$$

*where $\mu(f)$ and $\mu(f,g)$ are the Milnor numbers of $f$ and $(f,g)$ and $A_{\mathbb{C}}$ is defined as in the real case.*

The natural question that arises after this theorem is to ask if it i possible to get rid of the mod2 in the equality, namely to express:

$$\chi(C_\delta^\varepsilon \cap \{g \geq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}),$$

or:

$$\chi(C_\delta^\varepsilon) \pm \chi(C_\delta^\varepsilon \cap \{g = \alpha\}),$$

in terms of the signature of a bilinear symmetric form defined on $A_{\mathbb{R}}$. In general, as far as we know, this is still unknown and the question remains open. However, in some cases, the problem is solved. The strategy used is to find $n-k$ analytic function-germs such that $I = \langle f, \ldots, f_k; m_1, \ldots, m_{n-k}\rangle$ and to relate:

$$\chi(C_\delta^\varepsilon \cap \{g \geq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}),$$

or:

$$\chi(C_\delta^\varepsilon) \pm \chi(C_\delta^\varepsilon \cap \{g = \alpha\}),$$

to the Poincaré-Hopf index at 0 of the following map $H$:

$$\begin{array}{rccc} H & : & (\mathbb{R}^n, 0) & \to & (\mathbb{R}, 0) \\ & & x & \mapsto & (f_1(x), \ldots, f_k(x); m_1(x), \ldots, m_{n-k}(x)). \end{array}$$

Let us list the cases for which this strategy works. If $k = n-1$ then $f^{-1}(0)$ is a curve (or a point). Let $H = (f_1, \ldots, f_{n-1}, \frac{\partial(g, f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_n)})$. Then we have:

$$\# \left\{\text{branches of } f^{-1}(0) \mid g > 0\right\} - \# \left\{\text{branches of } f^{-1}(0) \mid g < 0\right\} = 2(-1)^{n-1}\mathrm{Ind}(H, 0).$$

Here a branch is a connected components of $f^{-1}(0) \setminus \{0\}$. This equality was proved in some cases by Aoki, Fukuda, Nishimura and Sun ([AFN1], [AFN2], [AFS]) and in full generality by Szafraniec [Sz2].

If $k = 1$ then $f$ is a function-germ with an isolated critical point at the origin. As already explained above, when $g = x_1$, Fukui [Fuk] proved that:

$$\chi(C_\delta^\varepsilon \cap \{x_1 \geq 0\}) - \chi(C_\delta^\varepsilon \cap \{x_1 \leq 0\}) = -\mathrm{sign}(-\delta)^n \mathrm{Ind}(H, 0),$$

where $H = (f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$. Note that $(\delta, 0)$ is always a regular value of $(f, x_1)$.

In [Du2], when $n = 2, 4$ or $8$, we were able to construct explicitly a map $H = (f, m_2, \ldots, m_n)$ such that:

$$\chi(C_\delta^\varepsilon \cap \{g \geq \alpha\}) - \chi(C_\delta^\varepsilon \cap \{g \leq \alpha\}) = -\mathrm{Ind}(H, 0).$$

The main idea that we used there is that the fibre $f^{-1}(\delta)$ is parallelizable (like the spheres $S^1 \subset \mathbb{R}^2$, $S^3 \subset \mathbb{R}^4$ and $S^7 \subset \mathbb{R}^8$). This last formula was extended by Fukui and Khovanskii in [FK]. They assume that $g$ satisfies the following Condition $(P)$: there exist $C^\infty$ vector fields $v_2, \ldots, v_n$ defined in a neighborhood $U$ of the origin such that:

1. $v_2(x), \ldots, v_n(x)$ span $T_x g^{-1}(g(x))$ whenever $\nabla g(x) \neq 0$,
2. $\det[\nabla g(x), v_2(x), \ldots, v_n(x)] > 0$.

Let $H$ be defined by:

$$\begin{array}{rcccc} H & : & (\mathbb{R}^n, 0) & \rightarrow & (\mathbb{R}^n, 0) \\ & & x & \mapsto & (f(x), v_1 f(x), \ldots, v_n f(x)). \end{array}$$

If 0 is isolated in $H^{-1}(0)$, then we have:

$$\chi(C_\delta^\varepsilon \cap \{g \geq 0\}) - \chi(C_\delta^\varepsilon \cap \{g \leq 0\}) = -\mathrm{sign}(-\delta)^n \mathrm{Ind}(H, 0),$$

if $(\delta, 0)$ is a regular value of $(f, g)$. If $n$ is even, we can replace $(\delta, 0)$ with a regular value $(\delta, \alpha)$ where $0 \leq |\alpha| \ll |\delta| \ll \varepsilon$. Furthermore, they gave situations where Condition $(P)$ is satisfied:

1. $n = 2, 4$ or $8$ (see [Du2]),
2. when $\frac{\partial g}{\partial x_1} \geq 0$,
3. if $\nabla g^{-1}(0) \cap B_\varepsilon^n \subset \{0\}$ then Condition $(P)$ is satisfied if:
   (a) $n = 2, 4$ or $8$,
   (b) or $n$ is even and $n \notin \{2, 4, 8\}$ and $\mathrm{Ind}(\nabla g, 0)$ is even,
   (c) or $n$ is odd and $\mathrm{Ind}(\nabla g, 0) = 0$.

In [Du10], we continued this work and made some improvements.

## 3.6 Global versions

In this section, we briefly report on global versions of the previous results.

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial map and let $W = F^{-1}(0)$. Let $G_1, \ldots, G_l : \mathbb{R}^n \to \mathbb{R}$ be polynomials. The problem is to compute the Euler-Poincaré characteristic of $W \cap \{G_1 ?_1 0, \ldots, G_l ?_l 0\}$ where $?_j \in \{<, \leq, \geq, >\}$ for $j \in \{1, \ldots, l\}$, i.e. to express it as a mapping degree or a signature.

If the dimension of the algebra $A = \frac{\mathbb{R}[x_1,\ldots,x_n]}{(F_1,\ldots,F_k)}$ is finite, then $W$ consists of a finite number of points. It is possible to express $\# W \cap \{G_1 ?_1 0, \ldots, G_l ?_l 0\}$ in terms of signatures of bilinear symmetric forms defined on $A$ (see [PRS] and [BW]).

In the case $W$ is a compact algebraic set, Szafraniec ([Sz1, Sz4]) and Bruce [Br] discovered a signature formula for $\chi(W)$. In [Du7], we extended it to semi-algebraic sets of the form $W \cap \{x_1 ?_1 0, \ldots, x_k ?_k 0\}$ where $W$ is compact, $k \in \{1, \ldots, k\}$ and $?_j \in \{\leq, \geq\}$.

The Bruce-Szafraniec method does not work if $W$ is not compact. In [Sz3, Sz5], Szafraniec proved several degree or signature formulas when $F = (F_1, \ldots, F_k)$ with $1 \leq k \leq n-1$ and $W = F^{-1}(0)$ is a smooth $(n-k)$-dimensional manifold (not necessarily compact). In [Du1, Du4] we gave formulas for some semi-algebraic sets of the form $W \cap \{G ? 0\}$, where $? \in \{\leq, \geq\}$, and of the form $W \cap \{G_1 ?_1 0, G_2 ?_2 0\}$, where $?_1$ and $?_2$ lie in $\{\leq, \geq\}$. In [Du3, Du8], we gave generalizations in some cases where $W$ admits isolated singularities. However, in general when $W$ is not compact, we do not have any signature formula for $\chi(W)$ and $\chi(W \cap \{G_1 ?_1 0, \ldots, G_l ?_l 0\})$. This seems to be a very difficult problem.

# Chapter 4

# Some results on the topology and geometry of semi-algebraic sets

In this chapter, we recall briefly the definition and the main properties of semi-algebraic sets and semi-algebraic maps. Then we give results on the topology and the geometry of semi-algebraic sets, that are singular versions of the results presented in Chapter 2. Our main references for the results of the first section of this chapter are [BR] and [BCR].

## 4.1 Definitions and important properties

**Definition 4.1.1** *A subset $V \subset \mathbb{R}^n$ is called semi-algebraic if its admits a representation of the form:*

$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_j} \{x \in \mathbb{R}^n \mid P_{i,j}(x)\sigma_{i,j}0\},$$

*where, for each $i = 1, \ldots, s$ and $j = 1, \ldots, r_j$:*

$$\sigma_{i,j} \in \{<, =, >\} \text{ and } P_{i,j} \in \mathbb{R}[x_1, \ldots, x_n].$$

Examples:

- Real algebraic sets are semi-algebraic.
- A semi-algebraic subset of $\mathbb{R}$ is either empty or a finite union of intervals (eventually reduced to a point or unbounded).

Let us list some important properties of semi-algebraic sets.

**Proposition 4.1.2** *1. The family of semi-algebraic sets is closed with respect to the set-theoretic operations of finite union, finite intersection and complementation.*

*2. A semi-algebraic set has a finite number of connected components and is locally connected.*

*3. (The Tarski-Seidenberg theorem) The projection of a semi-algebraic set is semi-algebraic.*

*4. The closure $\bar{X}$ of a semi-algebraic set $X$, its interior $\mathring{X}$ and its frontier $\bar{X} \setminus \mathring{X}$ are semi-algebraic.*

**Definition 4.1.3** *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic sets. A map $f : X \to Y$ is called semi-algebraic if its graph is a semi-algebraic set of $\mathbb{R}^{n+m}$.*

**Proposition 4.1.4** *Let $f : X \to Y$ be a semi-algebraic map. Then the image $f(X) \subset Y$ is a semi-algebraic set.*

**Theorem 4.1.5** *(Hardt's theorem) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two semi-algebraic sets and let $f : X \to Y$ be a semi-algebraic continuous map. There exists a finite partition of $Y$ into semi-algebraic sets $Y = \sqcup_{j=1}^r Y_j$ such that $f$ is semi-algebraically trivial over each $Y_j$. This means that there exists a semi-algebraic set $F_j$ and a semi-algebraic homeomorphism $h_j : f^{-1}(Y_j) \to Y_j \times F_j$ such that the following diagram commutes:*

$$\begin{array}{ccc} f^{-1}(Y_j) \subset X & \xrightarrow{h_j} & Y_j \times F_j \\ f \searrow & & \swarrow \text{projection} \\ & Y_j \subset Y & \end{array}$$

*Moreover if $Z_1, \ldots, Z_q$ are finitely many semi-algebraic subsets of $X$, we can ask that each trivialization $h_j : f^{-1}(Y_j) \to Y_j \times F_j$ is compatible with all the $Z_j$'s.*

**Theorem 4.1.6** *Every semi-algebraic set admits a semi-algebraic and finite Whitney stratification. This means that if $X \subset \mathbb{R}^n$ is a semi-algebraic set then there exists a finite semi-algebraic partition of $X$, $X = \sqcup_{j=1}^l S_j$, such that each $S_j$ is a smooth semi-algebraic manifold and this partition is a Whitney stratification of $X$.*

We end this section with an important result on Whitney stratified sets (not necessarily semi-algebraic). Let $X \subset \mathbb{R}^n$ be a closed Whitney stratified set and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map such that:

1. $f_{|X}$ is proper,
2. for each stratum $S$ of $X$, the restriction $f_{|S} : S \to \mathbb{R}^m$ is a submersion.

**Definition 4.1.7** *We call $f_{|X}$ a proper stratified submersion.*

**Theorem 4.1.8** *(Thom's first isotopy lemma) Let $f_{|X} : X \to \mathbb{R}^m$ be a proper stratified submersion. Then $f_{|X}$ is trivial i.e. there exists an homeomorphism $h : X \to \mathbb{R}^m \times f_{|X}^{-1}(0)$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f_{|X}"'] & & \mathbb{R}^m \times f_{|X}^{-1}(0 \arrow[dl, "\text{projection}"] \\ & \mathbb{R}^m & \end{tikzcd}$$

## 4.2 Integration with respect to the Euler characteristic and Poincaré-Hopf type theorems

In this section, we present Viro's method of integration with respect to the Euler characteristic with compact support (see [Vi]). We derive a Morse theory type theorem for semi-algebraic functions on semi-algebraic sets.

We first give the definition of the Euler characteristic with compact support, denoted by $\chi_c$. Our definition is specific to the semi-algebraic case and there are more general definitions. If $X \subset \mathbb{R}^n$ is a semi-algebraic set then it is possible to write it in the following way (see [BCR], Theorem 2.3.6):

$$X = \sqcup_{j=1}^{l} C_j,$$

where $C_j$ is semi-algebraically homeomorphic to $]-1,1[^{d_j}$ ($C_j$ is called a cell of dimension $d_j$). We set $\chi_c(X) = \sum_{j=1}^{l}(-1)^{d_j}$.

**Remark 4.2.1** *This definition of $\chi_c$ does not depend on the cell decomposition.*

**Proposition 4.2.2**
- *If $X$ is compact, then $\chi_c(X) = \chi(X)$,*
- *$\chi_c$ is multiplicative: $\chi_c(X \times Y) = \chi_c(X) \times \chi_c(Y)$,*

- $\chi_c$ *is additive:* $\chi_c(X \sqcup Y) = \chi_c(X) + \chi_c(Y)$,
- $\chi_c$ *is invariant by (semi-algebraic) homeomorphism.*

Examples: $\chi_c(\{*\}) = 1$, $\chi_c(\mathbb{R}) = -1$, $\chi_c([0, +\infty[) = 0$, $\chi_c(\mathbb{R}^2) = 2$, $\chi(\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}) = 1$ because the open first quadrant is the product $]0, +\infty[\times]0, +\infty[$.

**Remark 4.2.3** *The Euler characteristic with compact support is not invariant by homotopy.*

**Definition 4.2.4** *Let $X \subset \mathbb{R}^n$ be a semi-algebraic set. A constructible function $\varphi : X \to \mathbb{Z}$ is a $\mathbb{Z}$-valued function that can be written as a finite sum:*

$$\varphi = \sum_{i \in I} m_i 1_{X_i},$$

*where $X_i$ is a semi-algebraic subset of $X$.*

The sum and the product of two constructible functions on $X$ are again constructible. The set of constructible functions on $X$ is thus a commutative ring, denoted by $F(X)$.

**Definition 4.2.5** *If $\varphi \in F(X)$ then we set:*

$$\int_X \varphi d\chi_c = \sum_{i \in I} m_i \chi_c(X_i),$$

*where $\varphi = \sum_{i \in I} m_i 1_{X_i}$. The integral $\int_X \varphi d\chi_c$ is called the Euler integral of $\varphi$.*

**Definition 4.2.6** *Let $f : X \to Y$ be a continuous semi-algebraic map and let $\varphi : X \to \mathbb{Z}$ be a constructible function. The pushforward $f_*\varphi$ of $\varphi$ along $f$ is the function $f_*\varphi : Y \to \mathbb{Z}$ defined by:*

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi d\chi_c.$$

**Proposition 4.2.7** *The pushforward of a constructible function is a constructible function.*

*Proof.* Let us write $\varphi = \sum_{i \in I} m_i 1_{X_i}$. By Hardt's theorem, there is a finite semi-algebraic partition $Y = \sqcup_{j \in J} Y_j$ such that, over each $Y_j$, there is a semi-algebraic trivialization of $f$ compatible with the $X_i$'s. Hence, since for any $y \in Y$, $f_*\varphi(y)$ is equal to $\sum_{i \in I} m_i \chi_c(X_i \cap f^{-1}(y))$, we see that $f_*\varphi$ is constant on each $Y_j$. □

**Theorem 4.2.8** *(Fubini's theorem) Let $f : X \to Y$ be a continuous semi-algebraic map and let $\varphi$ be a constructible function on $X$. Then we have:*

$$\int_Y f_*\varphi d\chi_c = \int_X \varphi d\chi_c.$$

*Proof.* We keep the notations of the previous proof. Let $j \in J$ and $y_j \in Y_j$. Then, for every $i \in I$, $f^{-1}(Y_j) \cap X_i$ is semi-algebraically homeomorphic to $Y_j \times (f^{-1}(y_j) \cap X_i)$. Therefore, we have:

$$\int_Y f_*\varphi d\chi_c = \sum_{j \in J} \chi_c(y_j) f_*\varphi(Y_j) = \sum_{j \in J} \chi_c(Y_j) \sum_{i \in I} m_i \chi_c(f^{-1}(y_j) \cap X_i) =$$

$$\sum_{i \in I} m_i \sum_{j \in J} \chi_c(Y_j)\chi_c(f^{-1}(y_j) \cap X_i) = \sum_{i \in I} m_i \sum_{j \in J} \chi_c(f^{-1}(Y_j) \cap X_i) =$$

$$\sum_{i \in I} m_i \chi_c(X_i) = \int_X \varphi d\chi_c.$$

□

Let us give a nice application of this theory. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a finite semi-algebraic Whitney stratification: $X = \sqcup_{\alpha \in \Lambda} S_\alpha$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$-semi-algebraic function.

**Definition 4.2.9** *A point $p \in X$ is a critical point of $f_{|X}$ if it is a critical point of $f_{|S(p)}$, where $S(p)$ is the stratum that contains $p$.*

**Definition 4.2.10** *If $p$ is an isolated critical point of $f_{|X}$, we define the index of $f$ at $p$ by:*

$$\mathrm{ind}(f, X, p) = 1 - \chi\big(X \cap \{f = f(p) - \delta\} \cap B_\varepsilon^n(p)\big),$$

*where $0 < \delta \ll \varepsilon \ll 1$.*

**Theorem 4.2.11** *If $X$ is compact and $f_{|X}$ has a finite number of critical points $p_1, \ldots, p_k$ then:*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}(f, X, p_i).$$

*Proof.* For all $x \in X$, let $\varphi(x) = \chi_c\big(X \cap f^{-1}(x^-) \cap B^n_\varepsilon(x)\big)$ where $x^-$ is a regular value of $f$ close to $f(x)$ with $x^- \leq f(x)$. Note that $\varphi$ is constructible because $\varphi(x) = 1$ if $x \notin \{p_1, \ldots, p_k\}$.

Applying Fubini's theorem, we get:

$$\int_X \varphi(x) d\chi_c = \int_{\mathbb{R}} \left( \int_{f^{-1}(y)} \varphi(x) d\chi_c \right) d\chi_c.$$

For any $y \in \mathbb{R}$, let $y^-$ be a regular value of $f_{|X}$ close to $y$ with $y^- \leq y$. Let us denote by $q_1, \ldots, q_s$ the critical points of $f_{|X}$ lying in $f^{-1}(y)$. We have:

$$\chi_c\big(X \cap f^{-1}(y^-)\big) = \chi_c\big(X \cap f^{-1}(y^-) \setminus \cup_{i=1}^s B^n_\varepsilon(q_i)\big) + \sum_{i=1}^s \chi_c\big(X \cap f^{-1}(y^-) \cap B^n_\varepsilon(q_i)\big) =$$

$$\chi_c\big(X \cap f^{-1}(y) \setminus \cup_{i=1}^s B^n_\varepsilon(q_i)\big) + \sum_{i=1}^s \varphi(q_i) =$$

$$\chi_c\big(X \cap f^{-1}(y) \setminus \{q_1, \ldots, q_s\}\big) + \sum_{i=1}^s \varphi(q_i) =$$

$$\int_{X \cap f^{-1}(y) \setminus \{q_1, \ldots, q_s\}} \varphi(x) d\chi_c(x) + \sum_{i=1}^s \varphi(q_i) = \int_{f^{-1}(y)} \varphi(x) d\chi_c(x).$$

Since $X$ is compact, $f(X)$ is a compact subset of $\mathbb{R}$. Let us choose $]A, B]$ such that $f(X) \subsetneq ]A, B]$. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_l$ be the critical values of $f$. Let us write:

$$]A, B] = ]\alpha_0, \alpha_1] \cup ]\alpha_1, \alpha_2] \ \cup \ldots \cup \ ]\alpha_{l-1}, \alpha_l].$$

Since $\chi_c(]a, b]) = 0$ and $f_{|X \cap ]\alpha_j, \alpha_{j+1}[}$ is a trivial fibration, we obtain that:

$$\int_{\mathbb{R}} \big( \int_{f^{-1}(y)} \varphi(x) d\chi_c \big) d\chi_c = \int_{]A,B]} \chi_c\big(X \cap f^{-1}(y^-)\big) d\chi_c = 0,$$

and so,

$$\int_X \varphi(x) d\chi_c = 0.$$

But:

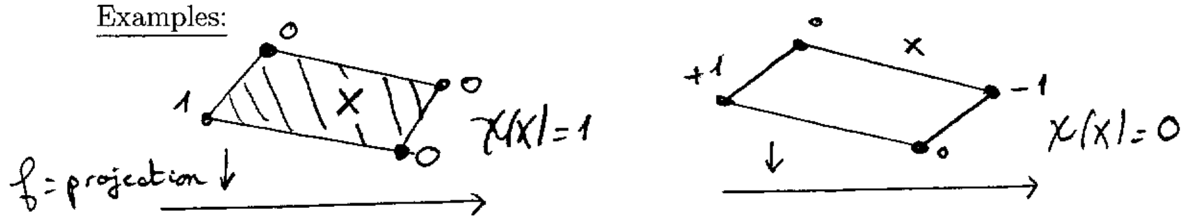
$$\int_X \varphi(x) d\chi_c = \chi_c(X \setminus \{q_1, \ldots, q_s\}) + \sum_{i=1}^s 1 - \mathrm{ind}(f, X, p_i),$$

and we find that:

$$0 = \chi(X) - \sum_{i=1}^s \mathrm{ind}(f, X, p_i).$$

□

Examples:




In a recent paper [Du12], we generalized this result to the case of closed semi-algebraic sets. Let us present these results now. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a semi-algebraic finite Whitney stratification $(S_\alpha)_{\alpha \in \Lambda}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ semi-algebraic function such that $f_{|X} : X \to \mathbb{R}$ has a finite number of critical points $p_1, \ldots, p_k$.

**Definition 4.2.12** *Let $* \in \{\leq, =, \geq\}$. We define $\Lambda_f^*$ by:*

$$\Lambda_f^* = \big\{\alpha \in \mathbb{R} \mid \beta \mapsto \chi\big(\mathrm{Lk}^\infty(X \cap \{f * \beta\})\big) \text{ is not constant} \\ \text{in a neighborhood of } \alpha\big\}.$$

Here $\mathrm{Lk}^\infty(Y) = Y \cap S_R^{n-1}$, $R \gg 1$, for any semi-algebraic set $Y$ of $\mathbb{R}^n$.

**Lemma 4.2.13** *The sets $\Lambda_f^{\leq}$, $\Lambda_f^{=}$ and $\Lambda_f^{\geq}$ are finite.*

We can write $\Lambda_f^{\leq} = \{a_1, \ldots, a_r\}$ where $a_1 < a_2 < \ldots < a_r$ and:

$$\mathbb{R} \setminus \Lambda_f^{\leq} = ]-\infty, a_1[\, \cup \,]a_1, a_2[\, \cup \cdots \cup \,]a_{r-1}, a_r[\, \cup \,]a_r, +\infty[.$$

On each connected component of $\mathbb{R} \setminus \Lambda_f^{\leq}$, the function $\beta \mapsto \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq \beta\}\big)$ is constant. For each $j \in \{0, \ldots, r\}$, let $a_j^+$ be an element of $]a_j, a_{j+1}[$ where $a_0 = -\infty$ and $a_{r+1} = +\infty$.

**Theorem 4.2.14** *We have:*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}(f, X, p_i) + \sum_{j=0}^{r} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq a_j^+\})\big) \\ - \sum_{j=1}^{r} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \leq a_j\})\big).$$

Similarly, we can write $\Lambda_f^{\geq} = \{b_1, \ldots, b_s\}$ with $b_1 < b_2 < \cdots < b_s$ and:

$$\mathbb{R} \setminus \Lambda_f^{\geq} = ]-\infty, b_1[ \cup ]b_1, b_2[ \cup \cdots \cup ]b_{s-1}, b_s[ \cup ]b_s, +\infty[.$$

For each $i \in \{0, \ldots, s\}$, let $b_i^+$ be an element in $]b_i, b_{i+1}[$ with $b_0 = -\infty$ and $b_{s+1} = +\infty$.

**Theorem 4.2.15** *We have:*

$$\chi(X) = \sum_{i=1}^{k} \mathrm{ind}(-f, X, p_i) + \sum_{j=0}^{s} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \geq b_j^+\})\big) - \sum_{j=1}^{s} \chi\big(\mathrm{Lk}^\infty(X \cap \{f \geq b_j\})\big).$$

Let us write $\Lambda_f^{=} = \{c_1, \ldots, c_t\}$ with $c_1 < c_2 < \ldots < c_t$ and:

$$\mathbb{R} \setminus \Lambda_f^{=} = ]-\infty, c_1[ \cup ]c_1, c_2[ \cup \cdots \cup ]c_{t-1}, c_t[ \cup ]c_t, +\infty[.$$

For each $i \in \{0, \ldots, t\}$, let $c_i^+$ be an element in $]c_i, c_{i+1}[$.

**Theorem 4.2.16** *We have:*

$$2\chi(X) - \chi\big(\mathrm{Lk}^\infty(X)\big) = \sum_{i=1}^{k} \mathrm{ind}(f, X, p_i) + \sum_{i=1}^{k} \mathrm{ind}(-f, X, p_i) + \sum_{j=0}^{t} \chi\big(\mathrm{Lk}^\infty(X \cap \{f = c_j^+\})\big) - \sum_{j=1}^{t} \chi\big(\mathrm{Lk}^\infty(X \cap \{f = c_j\})\big).$$

When $X = \mathbb{R}^n$, we find global versions of the Khimshiashvili formula and the Arnol'd-Wall formula presented in Chapter 2.

**Theorem 4.2.17** *We have:*

$$1 = \deg_\infty \nabla f + \sum_{j=0}^{r} \chi\big(\mathrm{Lk}^\infty(\{f \leq a_j^+\})\big) - \sum_{j=1}^{r} \chi\big(\mathrm{Lk}^\infty(\{f \leq a_j\})\big) =$$

$$(-1)^n \deg_\infty \nabla f + \sum_{j=0}^{s} \chi\big(\mathrm{Lk}^\infty(\{f \geq b_j^+\})\big) - \sum_{j=1}^{s} \chi\big(\mathrm{Lk}^\infty(\{f \geq b_j\})\big).$$

*If $n$ is even then we have:*

$$2 = 2\deg_\infty \nabla f + \sum_{j=0}^{t} \chi\big(\mathrm{Lk}^\infty(\{f = c_j^+\})\big) - \sum_{j=1}^{t} \chi\big(\mathrm{Lk}^\infty(\{f = c_j\})\big).$$

Here $\deg_\infty \nabla f$ is the degree of the map $\frac{\nabla f}{|\nabla f|} : S_R^{n-1} \to S^{n-1}$ where $S_R^{n-1}$ is a sufficiently big sphere.

**Remark 4.2.18** *The third equality of this last theorem was discovered by Sekalski [Se] for $n = 2$.*

## 4.3 Gauss-Bonnet type theorems

In Chapter 2 of this mini-course, we gave a Gauss-Bonnet formula for a smooth compact hypersurface $M \subset \mathbb{R}^n$. Here we present a version for smooth submanifolds of $\mathbb{R}^n$ and afterwards we give semi-algebraic versions of the Gauss-Bonnet theorem.

### 4.3.1 Smooth case

Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $d$ $(1 \leq d \leq n-1)$. Let $x \in M$ and let $S_x$ denotes the unit sphere in $(T_x M)^\perp$. Let $v \in S_x$ and let $II_{x,v}$ be the second fundamental form of $M$ at $x$ along the vector $v$. It is defined as follows:

$$II_{x,v}(x_1, x_2) = -\langle DV(x)(x_1), x_2 \rangle,$$

where:

- $V$ is a vector field in $\mathbb{R}^n$ normal to $M$ such that $V(x) = v$,
- $x_1, x_2 \in T_x M$.

The form $II_{x,v}$ is bilinear and symmetric.

**Definition 4.3.1** *For $i \in \{0, \ldots, d\}$ and for $x \in M$, we define $K_i(x)$ by:*

$$K_i(x) = \int_{S_x} \sigma_i(II_{x,v}) dv,$$

*where $\sigma_i$ is the $i$-th elementary symmetric function of the eigenvalues of $II_{x,v}$. We call the $K_i$'s the $i$-th Lipschitz-Killing curvatures.*

**Remark 4.3.2**
- *If $i$ is odd then $K_i(x) = 0$.*
- *The quantity $\frac{1}{s_{n-d+i-1}} K_i$ is intrinsic (here $s_k$ is the volume of $S^k$).*

The following Gauss-Bonnet theorem is due to Fenchel [Fe] and Allendoerfer [Al]

**Theorem 4.3.3** *(Gauss-Bonnet theorem) If $M$ is compact then:*

$$\chi(M) = \frac{1}{s_{n-1}} \int_M K_d(x)dx.$$

**Remark 4.3.4** *This theorem is trivial if $M$ is odd-dimensional because both sides of the equality vanish.*

The following theorem is due to Weyl [We].

**Theorem 4.3.5** *(Volume of the tube) If $r > 0$ is small enough, then:*

$$\mathrm{vol}(\mathrm{Tub}_r(M)) = \sum_{i=0}^{d} \frac{1}{n-d+i} \int_M K_i(x)dx \cdot r^{n-d+i}.$$

*Here $\mathrm{Tub}_r(M)$ is the tubular neighborhood of radius $r$ around $M$.*

Example Let $\mathcal{C} \subset \mathbb{R}^3$ be the circle centered at the origin and of radius $R$. Then $\mathrm{Tub}_r(\mathcal{C})$ is a torus. Applying the previous theorem, we obtain:

$$\mathrm{vol}(\mathrm{Tub}_r(\mathcal{C})) = \frac{1}{3-1} \int_{\mathcal{C}} \mathrm{Vol} S^1 dx r^2 = \frac{1}{2}(2\pi)(2\pi R)r^2.$$

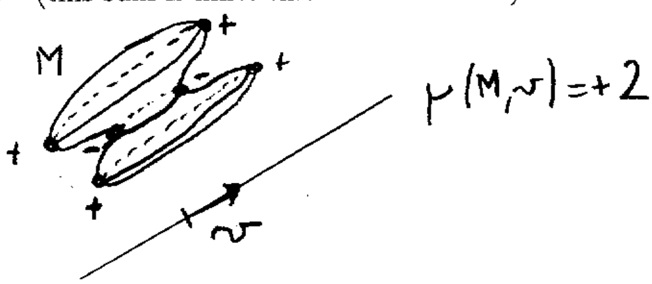
Hence we recover the well-known result:

$$\mathrm{vol}(\mathrm{Tub}_r(\mathcal{C})) = 2\pi^2 R r^2.$$

### 4.3.2 Exchange formulas

In this subsection, we explain how to give a topological proof of the Gauss-Bonnet theorem using Morse theory.

Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $d$ $(1 \leq d \leq n-1)$. For almost all $v \in S^{n-1}$, the function $v^*_{|M} : M \to \mathbb{R}$, $x \mapsto \langle v, x \rangle$ is a Morse function and hence admits isolated non-degenerate critical points $\{p_i\}_{i \in I}$, with respective indices $\{\lambda_i\}_{i \in I}$. Let $U \subset M$ be a bounded borelian set. We set $\mu(U, v) = \sum_{i \mid p_i \in U} (-1)^{\lambda_i}$ (this sum is finite since $U$ is bounded).



The following proposition is proved in [La].

**Proposition 4.3.6** *(Exchange formula) We have:*

$$\int_U K_d(x)dx = \int_{S^{n-1}} \mu(U,v)dv.$$

As a corollary, we recover the above Gauss-Bonnet formula.

**Corollary 4.3.7** *If $M$ is compact, then we have:*

$$\int_M K_d(x)dx = s_{n-1}\chi(M).$$

### 4.3.3 Singular semi-algebraic case

Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set equipped with a finite and semi-algebraic Whitney stratification: $X = \sqcup_{\alpha\in\Lambda} S_\alpha$.

**Lemma 4.3.8** *There exists a semi-algebraic set $\Gamma_1(X) \subset S^{n-1}$ of dimension strictly less than $n-1$ such that if $v \notin \Gamma_1(X)$, then $v^*_{|X}$ has a finite number of critical points $p_1^v, \ldots, p_{l_v}^v$.*

**Definition 4.3.9** *Let $U$ be a bounded borelian set of $X$. We set:*

$$\Lambda_0(X,U) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x\in U} \mathrm{ind}(v^*, X, x) dv,$$

*where $\mathrm{ind}(v^*, X, x) = 0$ if $x$ is not a critical point of $v^*_{|X}$. The measure $\Lambda_0(X,-)$ is called the Gauss-Bonnet measure.*

The following results are due to Broecker and Kuppe [BK] and Fu [Fu].

**Proposition 4.3.10**

1. *(Gauss-Bonnet theorem) If $X$ is compact, then we have:*

$$\Lambda_0(X,X) = \chi(X).$$

2. *The measure $\Lambda_0(X,-)$ is invariant by semi-algebraic isometries.*

Now we explain how to generalize the above Gauss-Bonnet theorem when $X$ is only closed (see [Du9] and [Du12])

**Lemma 4.3.11** *There exists a semi-algebraic set $\Gamma_2(X)$ of $S^{n-1}$ of dimension strictly less than $n-1$ such that if $v \notin \Gamma_2(X)$, $\Lambda^{\geq}_{v^*} = \Lambda^{\leq}_{v^*} = \Lambda^{=}_{v^*} = \emptyset$.*

**Corollary 4.3.12** *If $v \notin \Gamma_1(X) \cup \Gamma_2(X)$ then for all $\alpha \in \mathbb{R}$, we have:*

$$\chi\big(\mathrm{Lk}^\infty(X \cap \{v^* = \alpha\})\big) = 2\chi(X) - \chi\big(\mathrm{Lk}^\infty(X)\big) - \sum_{i=1}^{l_v} \mathrm{ind}(v^*, X, p_i) - \sum_{i=1}^{l_v} \mathrm{ind}(-v^*, X, p_i).$$

*Proof.* Apply Theorem 4.2.16. □

Let $(K_R)_{R>0}$ be an exhaustive family of compact sets of $X$, that is a family $(K_R)_{R>0}$ of compact sets of $X$ such that $\cup_{R>0} K_R = X$ and $K_R \subseteqq K_{R'}$ if $R \le R'$. For every $R > 0$, we have:

$$\Lambda_0(X, X \cap K_R) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X \cap K_R} \mathrm{ind}(v^*, X, x) dv.$$

Moreover the following limit:

$$\lim_{R \to +\infty} \sum_{x \in X \cap K_R} \mathrm{ind}(v^*, X, x),$$

is equal to $\sum_{x \in X} \mathrm{ind}(v^*, X, x)$ which is uniformly bounded by Hardt's theorem. Applying Lebesgue's theorem, we obtain:

$$\lim_{R \to +\infty} \Lambda_0(X, X \cap K_R) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \lim_{R \to +\infty} \sum_{x \in X \cap K_R} \mathrm{ind}(v^*, X, x) dv = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X} \mathrm{ind}(v^*, X, x) dv.$$

**Definition 4.3.13** *We set:*

$$\Lambda_0(X, X) = \lim_{R \to +\infty} \Lambda_0(X, X \cap K_R),$$

*where $(K_R)_{R>0}$ is an exhaustive family of compact sets of $X$.*

**Theorem 4.3.14** *If $X$ is a closed semi-algebraic set then:*

$$\Lambda_0(X, X) = \chi(X) - \frac{1}{2}\chi\big(\mathrm{Lk}^\infty(X)\big) - \frac{1}{2s_{n-1}} \int_{S^{n-1}} \chi\big(\mathrm{Lk}^\infty(X \cap \{v^* = 0\})\big) dv.$$

*Proof.* We have:

$$\Lambda_0(X,X) = \frac{1}{s_{n-1}} \int_{S^{n-1}} \sum_{x \in X} \mathrm{ind}(v^*, X, x) dv =$$

$$\frac{1}{2s_{n-1}} \int_{S^{n-1}} \sum_{x \in X} \mathrm{ind}(v^*, X, x) + \mathrm{ind}(-v^*, X, x) dv =$$

$$\frac{1}{2s_{n-1}} \int_{S^{n-1}} 2\chi(X) - \chi(\mathrm{Lk}^\infty(X)) - \chi(\mathrm{Lk}^\infty(X \cap \{v^* = 0\}))dv,$$

by Corollary 4.3.12. □

If $X$ is smooth of dimension $d$, $1 \leq d \leq n-1$, then:

$$\Lambda_0(X,X) = \frac{1}{s_{n-1}} \int_X K_d(x)dx.$$

If $d$ is even then $\chi(\mathrm{Lk}^\infty(X)) = 0$ because $\mathrm{Lk}^\infty(X)$ is a compact odd-dimensional manifold. Furthermore, $\mathrm{Lk}^\infty(X \cap \{v^* = 0\})$ is equal to $X \cap \{v^* = 0\} \cap S_R^{n-1}$ where $R \gg 1$ ; it is thus the boundary of a compact odd-dimensional manifold with boundary and therefore its Euler characteristic is equal to $\chi(X \cap \{v^* = 0\} \cap B_R^n)$, which is actually $\chi(X \cap \{v^* = 0\})$. Hence, if $d$ is even, the Gauss-Bonnet formula takes the following form:

$$\frac{1}{s_{n-1}} \int_X K_d(x)dx = \chi(X) - \frac{1}{s_{n-1}} \int_{S^{n-1}} \chi(X \cap \{v^* = 0\})dv.$$

Examples:

1. Let $V_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 - 1 = 0\}$ be the one-sheeted hyperboloid. We have $\int_{V_1} K_2(x)dx = -4\pi\sqrt{2}$.

2. Let $V_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 - 1 = 0\}$ be the two-sheeted hyperboloid. We have $\int_{V_2} K_2(x)dx = 4\pi(2 - \sqrt{2})$.

3. Let $V_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3 = 0\}$ be the elliptic paraboloid. We have $\int_{V_3} K_2(x)dx = 4\pi$.

4. Let $V_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3 = 0\}$ be the hyperbolic paraboloid. We have $\int_{V_4} K_2(x)dx = -4\pi$.

If $d$ is odd then:

$$\Lambda_0(X,X) = \frac{1}{s_{n-1}} \int_X K_d(x)dx = 0,$$

and,
$$\chi(\mathrm{Lk}^\infty(X \cap \{v^* = 0\}) = 0,$$
because $\mathrm{Lk}^\infty(X \cap \{v^* = 0\})$ is an odd-dimensional compact manifold . Furthermore for the same reasons as above, $\chi(X) = \frac{1}{2}\chi(\mathrm{Lk}^\infty(X))$. So, in case of an odd-dimensional closed semi-algebraic manifold, the above Gauss-Bonnet theorem is trivial, as in the compact case. However, the Euler characteristic of such a manifold is not necessarily zero and one can ask if it is possible to express it in terms of curvatures. This is actually the aim of the following theorem that we proved in [Du11].

**Theorem 4.3.15** *Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set which is a smooth submanifold of dimension $d$, $1 \le d \le n-1$. If $d$ is even, we have:*

$$\chi(X) = \frac{1}{s_{n-1}} \int_X K_d(x)dx + \sum_{i=0}^{\frac{d-2}{2}} \lim_{R \to +\infty} \frac{1}{s_{n-d+2i-1} b_{d-2i} R^{d-2i}} \int_{X \cap B_R} K_{2i} dx,$$

*where $b_i$ denotes the volume of the unit ball of dimension $i$. If $d$ is odd, we have:*

$$\chi(X) = \sum_{i=0}^{\frac{d-1}{2}} \lim_{R \to +\infty} \frac{1}{s_{n-d+2i-1} b_{d-2i} R^{d-2i}} \int_{X \cap B_R} K_{2i} dx.$$

Examples:

- If $d = 1$ then $X$ is a smooth semi-algebraic curve. The above formula just states that the number of non-compact connected components of $X$ is equal to $\lim_{R \to +\infty} \frac{\mathrm{length}(X \cap B_R^n)}{2R}$.
- If $X$ is of dimension 3, then the formula relates $\chi(X)$ to the volume form and the scalar curvature $K_2$. Namely, we have:
$$\chi(X) = \lim_{R \to +\infty} \frac{1}{s_{n-2} b_1 R} \int_{X \cap B_R} K_2 dx + \lim_{R \to +\infty} \frac{\mathrm{vol}(X \cap B_R^n)}{b_3 R^3}.$$
Let us give an application of this equality. If $K_2 > 0$ then $\chi(X) > 0$ and $\chi(\mathrm{Lk}^\infty(X)) > 0$. If the link $\mathrm{Lk}^\infty(X)$ is orientable then we can conclude that $\mathrm{Lk}^\infty(X)$ has at least one connected component homeomorphic to $S^2$.

Let us end with some remarks and questions.

1. A version of Theorem 4.3.14 was proved by Dillen and Kuehnel for submanifolds with finitely many cone-like ends in [DK].

2. A version of Theorem 4.3.15 was proved by Shiohama in [Sh] for a class of riemaninan surfaces (i.e. $d = 2$).

3. Is it possible to enlarge the class of riemannian manifolds for which a similar formula is valid ?

4. Is it possible to replace in Theorem 4.3.15 the distance on $\mathbb{R}^n$ with the intrinsic distance on $X$, in order to get a fully intrinsic formula ?

# Bibliography


[Al] ALLENDOERFER, C.B.: The Euler number of a Riemann manifold, *Amer. J. Math.* **62** (1940), 243-248.

[AFS] AOKI, K., FUKUDA, T., SUN, W.Z.: On the number of branches of a plane curve germ, *Kodai Math. Journal* **9** (1986), 179-187.

[AFN1] AOKI, K., FUKUDA, T., NISHIMURA. T.: On the number of branches of the zero locus of a map germ $(\mathbf{R}^n, 0) \to (\mathbf{R}^{n-1}, 0)$. *Topology and Computer Science : Proceedings of the Symposium held in honor of S. Kinoshita, H. Noguchi and T. Homma on the occasion of their sixtieth birhtdays* (1987), 347-363.

[AFN2] AOKI, K., FUKUDA, T., NISHIMURA. T.: An algebraic formula for the topological types of one parameter bifurcation diagrams, *Archive for Rational Mechanics and Analysis* **108** (1989), 247-265.

[AGV] ARNOLD, V.I., GUSEIN-ZADE, S.M., VARCHENKO, A.N.: Singularities of differentiable maps, (vol.1) *Birkhauser* (1988).

[Ar] ARNOL'D, V.I.: Index of a singular point of a vector field, the Petrovski-Oleinik inequality, and mixed Hodge structures, *Funct. Anal. and its Appli.* **12** (1978), 1-14.

[ADD] ARAÚJO dos SANTOS, R., DREIBELBIS, D., DUTERTRE, N.: Topology of the real Milnor fiber for isolated singularities, *Real and Complex Singularities, Contemporary Mathematics*, **569** (2012), 67-75.

[BCRS] BECKER, E., CARDINAL, J.P., ROY, M.F., SZAFRANIEC, S.: Multivariate Bezoutians, Kronecker symbol and Eisenbud & Levine formula, *Algorithms in Algebraic Geometry and Applications* Progress in Mathematics **143** , 79-104 Birkhauser (1996).

[BW] BECKER, E., WOERMANN, T.: On the trace formula for quadratic forms. *Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991)*, 271-291, Contemp. Math., **155**, Amer. Math. Soc., Providence, RI, 1994.

[BR] BENEDETTI, R., RISLER, J.J.: Real algebraic and semi-algebraic sets, Hermann (1990).

[BCR] BOCHNAK, J., COSTE, M., ROY, M.F.: Géométrie algébrique réelle, *Ergebnisse der Mathematik* **12** Springer-Verlag (1987).

[BK] BROECKER, L., KUPPE, M.: Integral geometry of tame sets, *Geometriae Dedicata* **82** (2000), 285-323.

[Br] BRUCE, J.W. : Euler characteristics of real varieties, *Bull. London Math. Soc.* **22** (1990), 547-552.

[DK] DILLEN, F., KÜHNEL, W.: Total curvature of complete submanifolds of Euclidean space, *Tohoku Math. J. (2)* **57** (2005), no. 2, 171-200.

[Du1] DUTERTRE, N.: An algebraic formula for the Euler characteristic of some semi-algebraic sets, *Journal of Pure and Applied Algebra* **139** (1999), 41-60.

[Du2] DUTERTRE, N.: Degree formulas for a topological invariant of bifurcations of function germs, *Kodai Mathematical Journal* **23**, no. 3 (2000), 442-461.

[Du3] DUTERTRE, N.: On affine complete intersection with isolated singularities, *Journal of Pure and Applied Algebra* **164**, no. 1-2 (2001), 129-147.

[Du4] DUTERTRE, N.: About the Euler-Poincaré characteristic of semi-algebraic sets defined with two inequalities, *Revista Matemática Complutense* **14**, no. 1 (2001), 45-82.

[Du5] DUTERTRE, N.: On the Milnor fibre of a real map-germ, *Hokkaido Mathematical Journal* **31** (2002), 301-319.

[Du6] DUTERTRE, N. : Courbures et singularités réelles, *Commentarii Mathematici Helvetici* **77**, no. 4 (2002), 846-863.

[Du7] DUTERTRE, N.: On the Euler-Poincaré characteristic of semi-analytic sets and semi-algebraic sets, *Math. Proc. Camb. Phil. Soc.* **135**, no. 3 (2003), 527-538.

[Du8] DUTERTRE, N.: On topological invariants associated with a polynomial with isolated critical points, *Glasg. Math. J.* **46**, no. 2 (2004), 323-334.

[Du9] DUTERTRE, N.: A Gauss-Bonnet formula for closed semi-algebraic sets, *Advances in Geometry* **8**, no.1 (2008), 33-51.

[Du10] DUTERTRE, N.: On the Euler characteristics of real Milnor fibres of partially parallelizable maps of $\mathbb{R}^n$ to $\mathbb{R}^2$, *Kodai Mathematical Journal* **32**, no. 2 (2009), 324-351.

[Du11] DUTERTRE, N.: Euler characteristic and Lipschitz-Killing curvatures of closed semi-algebraic sets, *Geometriae Dedicata* **158** (2012), no. 1,167-189.

[Du12] DUTERTRE, N.: On the topology of semi-algebraic functions on closed semi-algebraic sets, *Manuscripta Mathematica* (2012), DOI: 10.1007/s00229-011-0523-0.

[Ei] EISENBUD, D.: An algebraic approach to the topological degree of a smooth map, *Bull. Amer. Math. Soc.* **84** (1978), no. 5, 751-764.

[EL] EISENBUD, D., LEVINE, H.I.: An algebraic formula for the degree of a $C^\infty$ map-germ, *Annals of Mathematics* **106** (1977), 19-44.

[Fe] FENCHEL, W.: On total curvature of riemannian manifolds I, *Journal of London Math. Soc.* **15** (1940), 15-22.

[Fu] FU, J.H.G.: Curvature measures of subanalytic sets, *Amer. J. Math.* **116** (1994), no. 4, 819-880.

[Fuk] FUKUI, T.: An algebraic formula for a topological invariant of bifurcation of 1-parameter family of function-germs, in *Stratifications, singularities, and differential equations, II (Marseille, 1990; Honolulu, HI, 1990)*, Travaux en cours **55**, Hermann, Paris, 45-54 1997.

[FK] FUKUI, T., KHOVANSKII, A.: Mapping degree and Euler characteristic, *Kodai Math. J.* **29**, no. 1 (2006), 144-162.

[GG] GOLUBITSKY, M., GUILLEMIN, V.: Stable mappings and their singularities, *Graduate Texts in Mathematics*, vol. 14, Springer-Verlag, New York-Heidelberg, 1973.

[Gr] GREUEL, G.M.: Der Gauss-Manin Zusammenhang isolierter Singularitäten von vollständingen Durschnitten, *Math. Annalen* **214** (1975), 235-266.

[GP] GUILLEMIN, V., POLLACK, A.: Differential topology, *Prentice-Hall, Inc., Englewood Cliffs, N.J.*, 1974.

[Ha] HAEFLIGER, A.: Quelques remarques sur les applications différentiables d'une surface dans le plan, *Ann. Inst. Fourier* **10** (1960), 47-60.

[Hi] HIRSCH, M.: Differential topology, *Graduate Texts in Mathematics*, no. 33, Springer-Verlag, New York-Heidelberg, 1976.

[Ho] HOPF, H.: Differential Geometry in the Large: Seminar Lectures NYU 1946 and Stanford 1956, *Lecture Notes in Mathematics*, Springer-Verlag **1000**, (1983).

[Kh] KHIMSHIASHVILI, G.M.: On the local degree of a smooth map, *Soobshch. Akad. Nauk Gruz. SSR* **85** (1977), 309-311.

[La] LANGEVIN, R.: Courbures, feuilletages et surfaces, Dissertation, Université Paris-Sud, Orsay, 1980. *Publications Mathématiques d'Orsay 80, 3.* Université de Paris-Sud, Département de Mathématiques, Orsay 1980.

[Le] LE DUNG TRANG: Calcul du nombre de Milnor d'une singularité isolée d'intersection complète, *Funct. Anal. Appl.* **8** (1974), 45-52.

[Mi1] MILNOR, J.: Topology from the differentiable viewpoint, *The University Press of Virginia, Charlottesville, Va.* 1965.

[Mi2] MILNOR, J.: Morse theory, *Ann. Math. Stud.* **51**, Princeton University Press (1963).

[Mi3] MILNOR, J.: Singular points of complex hypersurfaces, *Ann. Math. Stud.* **61**, Princeton University Press (1968).

[PRS] PEDERSEN, P., ROY, M.-F., SZPIRGLAS, A.: Counting real zeros in the multivariate case. *Computational algebraic geometry (Nice, 1992)*, 203-224, Progr. Math., **109**, Birkhauser Boston, Boston, MA, (1993).

[Sa] SAMELSON, H.: On immersion of manifolds, *Canadian J. Math* **12** (1960), 529-534.

[Se] SEKALSKI, M.: The degree at infinity of the gradient of a polynomial in two real variables, *Ann. Polon. Math.* **87** (2005), 229-235.

[Sh] SHIOHAMA, K.: Total curvatures and minimal areas of complete surfaces, *Proc. Amer. Math. Soc.* **94** (1985), no. 2, 310-316.

[Sz1] SZAFRANIEC, Z. : On the Euler characteristic of analytic and algebraic sets, *Topology* **26** (1986), 411-414.

[Sz2] SZAFRANIEC, Z. : On the number of branches of a 1-dimensional semi-analytic set, *Kodai Math. Journal* **11** (1988), 78-85.

[Sz3] SZAFRANIEC, Z. : The Euler characteristic of algebraic complete intersections, *Jour. reine angew Math.* **397** (1989), 194-201.

[Sz4] SZAFRANIEC, Z. : Topological invariants of weighted homogeneous polynomials, *Glascow Math. Journal* **33**, (1991) 241-245.

[Sz5] SZAFRANIEC, Z. : A formula for the Euler characteristic of a real algebraic manifold, *Manuscripta Mathematica* **85** (1994), 345-360.

[Vi] VIRO, O. : Some integral calculus based on Euler characteristic, *Topology and geometry, Rohlin Seminar*, 127-138, Lecture Notes in Math., 1346, Springer, Berlin, 1988.

[Wa] WALL, C.T.C.: Topological invariance of the Milnor number mod 2, *Topology* **22** (1983), 345-350.

[We] WEYL, H.: On the volume of tubes, *Amer. J. Math.* **61** (1939), 461-472.